1.4 Squeeze Theorem

Standards:

MCA2

MCA2a

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$\boxed{\text{old}}$ Calculating Limits Algebraically

① $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1} \dfrac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim\limits_{x\to 1} x+1 = 1+1 = 2.$

② $\lim\limits_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7} = \dfrac{\sqrt{7+2}-3}{7-7} = \dfrac{0}{0}$

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x\to 7} \frac{(x+2)-3\sqrt{x+2}+3\sqrt{x+2}-9}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{x+2-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{\cancel{x-7}}{\cancel{(x-7)}(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{1}{\sqrt{x+2}+3}$$

$$= \frac{1}{\sqrt{7+2}+3} = \frac{1}{3+3} = \frac{1}{6}.$$

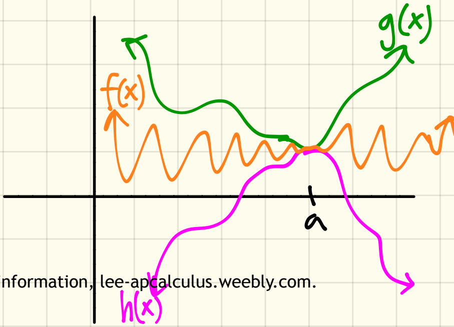
$\boxed{\text{new}}$ Squeeze Theorem

Let's consider $\lim\limits_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Sometimes we cannot find limits directly (by algebraic manipulation), so the squeeze theorem (or "sandwich theorem") can help.

Suppose we have 2 functions $g(x)$ & $h(x)$ represented graphically.

Also, let's say there is a function $f(x)$ that is "sandwiched between the 2 functions



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$\boxed{\text{Squeeze Theorem}}$

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq a$ in some interval about a, and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L,$$

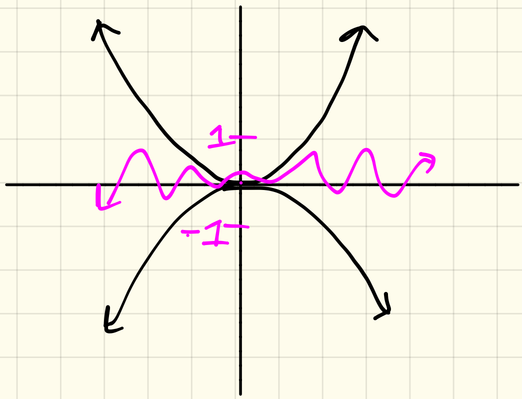
then $\lim_{x\to a} f(x) = L$.

Going back to the previous situation.

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

note: we know that values of sine function lie between -1 and 1.

$$\underbrace{\lim_{x\to 0} x^2}_{\text{use this!}} \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$



$$-1 \leq x^2 \sin\left(\frac{1}{x}\right) \leq 1$$

$$-1x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq 1x^2$$

$\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$

So, the squeeze theorem gives $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

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[Example 1] If $3x \le f(x) \le x^3+2$ for $0 \le x \le 2$, evaluate $\lim_{x\to 1} f(x)$.

$$\lim_{x\to 1} 3x = 3(1) = 3$$

$$\lim_{x\to 1} x^3+2 = (1)^3+2 = 3$$

Since both limits equal 3, then applying the Squeeze Theorem gives that $\lim_{x\to 1} f(x) = 3$.

[Example 2] Find $\lim_{x\to 1} f(x)$ given that $4 \le f(x) \le x^2+6x-3$ for all x.

$$\lim_{x\to 1} 4 = 4$$

$$\lim_{x\to 1} x^2+6x-3 = (1)^2+6(1)-3 = 4$$

So, $\lim_{x\to 1} f(x) = 4$ when the squeeze theorem is applied.

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