2.10 Logarithmic Differentiation

Standard:

MCD1e

$\boxed{\text{old}}$ Properties of Logarithmic & Exponential Functions

Let's recall properties of exponential functions:

① $e^a \cdot e^b = e^{a+b}$ ② $\dfrac{e^a}{e^b} = e^{a-b}$ ③ $(e^a)^b = e^{ab}$

④ $e^{-n} = \dfrac{1}{e^n}$ ⑤ $e^0 = 1$.

Let's recall properties of logarithmic functions:

① $\ln x + \ln y = \ln(xy)$ ② $\ln x - \ln y = \ln\left(\dfrac{x}{y}\right)$

③ $\ln x^y = y \ln x$ ⑥ $\ln 1 = 0$

$\boxed{\text{new}}$ Logarithmic Differentiation

Let's consider $y = x^n$. (We know that $\frac{d}{dx} x^n = nx^{n-1}$.) Let's assume that we don't know the power rule, but we want the formula for it.

So, $y = x^n$.

Then, $\ln y = \ln x^n$ — take ln on both sides

$\ln y = n \ln x$ — simplify using law of logs

$\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(n \ln x)$ — differentiate

$\dfrac{1}{y} \cdot y' = n \cdot \dfrac{1}{x}$ — implicit differentiation with respect to x

$$y' = n \cdot \frac{y}{x}$$ — solve for y'

$$y' = n \cdot \frac{x^n}{x}$$ — fact: $y = x^n$

$$y' = n x^{n-1}. \blacksquare$$

[Examples] Differentiate.

① $f(x) = x^x$

Let $y = x^x$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y} \cdot y' = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\frac{1}{y} \cdot y' = 1 + \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x).$$

② $y = (\sin x)^{\ln x}$

$$\ln y = \ln(\sin x)^{\ln x}$$

$$\ln y = \ln x\,[\ln(\sin x)]$$

$$\frac{1}{y} \cdot y' = \ln x \cdot \frac{1}{\sin x} \cdot \cos x + \ln(\sin x) \cdot \frac{1}{x}$$

$$\frac{1}{y} \cdot y' = \frac{\ln x \cos x}{\sin x} + \frac{\ln(\sin x)}{x}$$

$$y' = y\left[\ln x \cot x + \frac{\ln(\sin x)}{x}\right]$$

$$y' = (\sin x)^{\ln x}\left[\ln x \cot x + \frac{\ln(\sin x)}{x}\right]$$