2.3 Derivatives Using Limits

Standards:

MCD1a

MCD1b

MCD1c

$\boxed{\text{Old}}$ Calculating Tangent Lines

To find the slopes of tangent lines, we could use the following formula:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ if limit exists.}$$

Find the IROC at $x = 1$ for $f(x) = 4 - x^2$.

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[4 - (1+h)^2] - [3]}{h}$$

$$= \lim_{h \to 0} \frac{[4 - (1+2h+h^2)] - [3]}{h}$$

$$= \lim_{h \to 0} \frac{4 - 1 - 2h - h^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} - 2h - h^2 - \cancel{3}}{h}$$

$$= \lim_{h \to 0} \frac{-2h - h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2 - h)}{\cancel{h}}$$

$$= \lim_{h \to 0} -2 - h$$

$$= -2 - (0) = -2$$

$\boxed{\text{new}}$ Derivatives

- are a way to measure the IROC of a function at any point
- gives us a function of x at any given point P(x, f(x)) which we will notate as $f'(x) \rightarrow$ "f prime of x"

- This "new" function gives the slope of the tangent line to the graph of f at the point (x, f(x)), provided that the graph has a tangent line at this point.

Definition

The derivative of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists. For all of (x) for which this limit exists, f' is the function of x.

Other notations for derivatives:

$\frac{d}{dx} f(x)$, y', $f'(x)$

[Example 1] Find the derivative of $f(x) = 4 - x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[4 - (x+h)^2] - [4 - x^2]}{h}$$

$$= \lim_{h \to 0} \frac{[4 - (x^2 + 2xh + h^2)] - [4 - x^2]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} \cancel{- x^2} - 2xh - h^2 \cancel{- 4} \cancel{+ x^2}}{h}$$

$$= \lim_{h \to 0} \frac{-2xh - h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x - h)}{\cancel{h}}$$

$$= \lim_{h \to 0} -2x - h$$

$$= -2x - (0)$$

$$= -2x.$$

The derivative of $f(x) = 4 - x^2$ is $f'(x) = -2x$.

How can I use the derivative to get the slopes of tangent lines?

$f(x) = 4 - x^2$

$f'(x) = -2x$

- Find the IROC at $x = 1$.

  $f'(1) = -2(1) = -2$

- Find the IROC at $x = 3$

  $f'(3) = -2(3) = -6$

Conclusion We can Now use the derivative of that function, to find the slopes of tangent lines (IROC) at any x-value, provided the slope of the tangent line exist at the indicated x-value.

[Example 2] Find $\frac{dy}{dx}$ for $f(x) = 2x^2$.

[Example 3] Find the slopes of the tangent each indicated point of the function: $f(x) = \sqrt{x}$.

ⓐ $x = 3$ ⓑ $x = 5$ ⓒ $x = 7$ ⓓ $x = 2$

[Example 4] What is the derivative of $f(x) = 3x^3$.

[Example 5] $\frac{dy}{dx}\left(\sqrt{1+x}\right) = ?$

[Example 6] Find the equation of the tangent lines for each of those slopes in Example 3.

[Example 2] $\frac{dy}{dx}(2x^2) = ?$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[2(x+h)^2] - [2x^2]}{h}$$

$$= \lim_{h \to 0} \frac{[2(x^2+2xh+h^2)] - [2x^2]}{h}$$

$$= \lim_{h \to 0} \frac{[2x^2+4xh+2h^2] - [2x^2]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x+2h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 4x + 2h = 4x + 2(0)$$

$$= 4x$$

[Example 3] $f(x) = \sqrt{x}$; $f'(x) = ?$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

(a) $f'(3) = \frac{1}{2\sqrt{3}}$

(b) $f'(5) = \frac{1}{2\sqrt{5}}$

(c) $f'(7) = \frac{1}{2\sqrt{7}}$

(d) $f'(2) = \frac{1}{2\sqrt{2}}$

[Example 4] $y = 3x^3$. what is y'?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x+h)^3] - [3x^3]}{h}$$

$$= \lim_{h \to 0} \frac{[3(x^3 + 3x^2h + 3xh^2 + h^3)] - [3x^3]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x^3}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2 + 3x(0) + (0)^2$$

$$= 3x^2$$