2.6 Chain Rule

Standards:

MCD2

MCD2a

This was created by Keenan Xavier Lee, 2013. See my website for more information, lee-apcalculus.weebly.com.

$\boxed{\text{Old}}$ Composition of Functions

Let's consider the functions: $f(x) = 2x+1$ and $g(x) = x^3$. Perform the following tasks.

1. $f \circ g(x) = f(g(x)) = 2(x^3) + 1 = 2x^3 + 1$
2. $g \circ f(x) = g(f(x)) = (2x+1)^3$
3. $f \circ f(x) = f(f(x)) = 2(2x+1) + 1 = 4x + 2 + 1 = 4x + 3$
4. $g \circ g(x) = g(g(x)) = (x^3)^3 = x^9$

Are any of the following functions a composition of functions? If so, state the composition.

1. x^3

   no

2. $(x^2+5)^3$

   $f(x) = x^2+5$
   $g(x) = x^3$

3. $\sqrt{x^3}$

   $f(x) = x^3$
   $g(x) = \sqrt{x}$

4. $\sin(4x)$

   $f(x) = \sin(x)$
   $g(x) = 4x$

5. $\cos^3 x$

   (rewrite) $= (\cos x)^3$
   $f(x) = \cos x$
   $g(x) = x^3$

6. $\dfrac{1}{(t^4+1)^3}$

   $f(x) = \dfrac{1}{x^3}$
   $g(x) = t^4+1$

7. $\tan(\sin x)$

   $f(x) = \tan x$
   $g(x) = \sin x$

8. $\cos(nx)$

   $f(x) = nx$
   $g(x) = nx$

This was created by Keenan Xavier Lee, 2013. See my website for more information, lee-apcalculus.weebly.com.

$\boxed{\text{new}}$ Chain Rule

We've been able to find derivatives of functions such as $f(x) = x^2$ & $g(x) = \sin(x)$. However, what if we wanted to differentiate such functions like $f(x) = \sqrt{x^2+1}$ and $g(x) = (x^3 + \tan x)^{76}$?

We need a technique to help us determine the derivative of the composition of functions.

Let's note...

Since functions $f(x) = \sqrt{x^2+1}$ & $g(x) = (x^3 + \tan x)^{76}$ are compositions, we can use a technique called the <u>chain rule</u>.

Let's consider $K(x) = f(g(x))$. Find the derivative.

$$\frac{dK(x)}{dx} \Longrightarrow \text{need to find derivative.}$$

$$\frac{dK(x)}{dx} \cdot \frac{dg(x)}{g(x)} = \frac{dK(x)}{d\,g(x)} \cdot \frac{dg(x)}{dx} = \frac{d\,f(g(x))}{d\,g(x)} \cdot \frac{dg(x)}{x}$$

multiply by 1.

$$= f'(g(x)) \cdot g'(x)$$

$\boxed{\text{CHAIN RULE}}$

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{derivative of the outside function}} \cdot \underbrace{g'(x)}_{\text{derivative of the inside function}}$$

This was created by Keenan Xavier Lee, 2013. See my website for more information, lee-apcalculus.weebly.com.

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

What is the outside & inside function of $f(x) = \sqrt{x^2+1}$ & $g(x) = (x^3 + \tan x)^{76}$?

$\boxed{a}$ $f(x) = \sqrt{x^2+1}$
- outside function: $f(x) = \sqrt{x}$
- inside function: $g(x) = x^2+1$

$\boxed{b}$ $g(x) = (x^3 + \tan x)^{76}$
- outside function: $f(x) = x^{76}$
- inside function: $g(x) = x^3 + \tan x$

[Examples] Find the derivatives of the functions.

① $f(x) = (x^3 + \tan x)^{76}$

$$f'(x) = 76(x^3 + \tan x)^{35} \cdot (3x^2 + \sec^2 x)$$

② $f(x) = \sqrt{x^2+1}$ Rewrite $= (x^2+1)^{1/2}$

$$f'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x)$$

$$= x(x^2+1)^{-1/2}$$

$$= \frac{x}{(x^2+1)^{1/2}}$$

$$= \frac{x}{\sqrt{x^2+1}}$$

This was created by Keenan Xavier Lee, 2013. See my website for more information, lee-apcalculus.weebly.com.

③ $\frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot(2x)$

$= 2x\cos(x^2)$

④ $f(x) = \sin^2 x$ Rewrite $(\sin x)^2$

$f'(x) = 2\sin x \cdot \cos x$

$= 2\sin x\cos x$

⑤ $f(x) = (x^3 - 1)^{100}$

$f'(x) = 100(x^3-1)^{99} \cdot (3x^2)$

$= 300x^2(x^3-1)^{99}$

⑥ $f(x) = \left(\frac{t-2}{2t+1}\right)^9$

$f'(x) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \left(\frac{(1)(2t+1)-(2)(t-2)}{(2t+1)^2}\right)$

This was created by Keenan Xavier Lee, 2013. See my website for more information, lee-apcalculus.weebly.com.