2.9 Derivatives of Logarithmic Functions

Standard:

MCD1e

$\boxed{\text{Old}}$ Exponential Derivatives & Implicit Differentiation

1. $x^2y = 7x$

$(x^2)(y') + (y)(2x) = 7$

$x^2y' + 2xy = 7$

$x^2y' = 7 - 2xy$

$y' = \dfrac{7-2xy}{x^2}$

2. $y = e^{\cos x}$

$y' = e^{\cos x} \cdot (-\sin x)$

$= -\sin x \, e^{\cos x}$

3. $e^{xy} = 7x$

$e^{xy} \cdot \left[(x)(y') + (y)(1)\right] = 7$

$e^{xy}\left[xy' + y\right] = 7$

$\dfrac{\cancel{e^{xy}}\left[xy' + y\right]}{\cancel{e^{xy}}} = \dfrac{7}{e^{xy}}$

$xy' + y = \dfrac{7}{e^{xy}}$

$xy' = \dfrac{7}{e^{xy}} - y$

$y' = \dfrac{\frac{7}{e^{xy}} - y}{x}$.

$\boxed{\text{new}}$ Logarithmic Derivatives

Let's consider $f(x) = \log_a x$. Let's find the derivative.

So $y = \log_a x \overset{\text{can be rewritten as}}{\Longleftrightarrow} a^y = x$.

Now let's differentiation.

$$a^y = x$$

$$a^y \cdot \ln(a) \cdot y' = 1$$

$$y' = \frac{1}{(\ln a)\, a^y}$$

because $a^y = x$,

$$y' = \frac{1}{(\ln a)\, x}$$

But really ... its $\frac{d}{dx} \log_a f(x) = \frac{1}{(\ln a) f(x)} \cdot f'(x)$

[Examples] Find the derivatives.

① $f(x) = \log_2 (1-3x)$

$$f'(x) = \frac{1}{(\ln 2)(1-3x)} (-3)$$

$$= \frac{-3}{(\ln 2)(1-3x)}$$

② $\frac{d}{dx}\left(\log_5 (xe^x)\right) =$

$$= \frac{1}{(\ln 5)(xe^x)} \cdot [(x)(e^x) + (e^x)(1)]$$

$$= \frac{xe^x + e^x}{(\ln 5)(xe^x)}$$

Now let's consider $f(x) = \ln x$. We want to find the derivative.

So, $y = \ln x \xleftrightarrow{\text{can be rewritten as}} e^y = x$

Now let's differentiation $e^y = x$.

$$e^y = x$$
$$e^y \cdot y' = 1$$
$$y' = \frac{1}{e^y}$$

because $e^y = x$,

$$= \frac{1}{x}.$$

But really ... its $\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x)$

Logarithmic Rules for Derivatives

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad \frac{d}{dx} \log_a x = \frac{1}{(\ln a)(x)}$$

[Examples] Find the derivatives.

① $y = \ln(\cos x)$

$$y' = \frac{1}{\cos x} \cdot (-\sin x)$$
$$= \frac{-\sin x}{\cos x}$$
$$= -\tan x.$$

② $\frac{d}{dx}(\ln(x^2+10)) =$

$$= \frac{1}{x^2+10} \cdot (2x)$$
$$= \frac{2x}{x^2+10}$$

This was created by Keenan Xavier Lee, 2013. See my website for more information, lee-apcalculus.weebly.com.

[More Examples] Differentiate problems.

① $f(x) = \ln(\sqrt[5]{x}) = \ln(x^{1/5})$

$f'(x) = \ln(x^{1/5})$

$= \dfrac{1}{x^{1/5}} \cdot \dfrac{1}{5}x^{-4/5}$

$= \dfrac{1}{5x^{1/5} \cdot x^{4/5}}$

$= \dfrac{1}{5x^{5/5}}$

$= \dfrac{1}{5x}$

② $f(x) = \log_6(2x-3x^2)$

$f'(x) = \dfrac{1}{(\ln 6)(2x-3x^2)}(2-6x)$

$= \dfrac{2-6x}{(\ln 6)(2x-3x^2)}$

③ $y = \cos(\ln x)$

$y' = -\sin(\ln x) \cdot \dfrac{1}{x}$

$= -\dfrac{\sin(\ln x)}{x}$

④ $y = \ln(x^4 \sin x)$

$y' = \dfrac{1}{x^4 \sin x} \cdot [(x^4)(\cos x) + (\sin x)(4x^3)]$

$= \dfrac{x^4 \cos x + 4x^3 \sin x}{x^4 \sin x}$

$= \dfrac{x^3(x\cos x + 4\sin x)}{x^4 \sin x}$

$= \dfrac{x\cos x + 4\sin x}{x \sin x}$

$= \dfrac{x\cos x}{x \sin x} + \dfrac{4\sin x}{x\sin x}$

$= \dfrac{\cos x}{\sin x} + \dfrac{4}{x} = \cot x + \dfrac{4}{x}$