4.1 Antiderivatives

Standards:

MC11

MC11b

$\boxed{\text{Old}}$ Derivatives $\longrightarrow$ Differentiation

Find the derivatives of each.

① $f(x) = x^4 + 3x^2 + 6x$

$f'(x) = 4x^3 + 8x + 6$

② $y = \ln(2x)$

$y' = \frac{1}{2x}(2) = \frac{1}{x}$

③ $g(x) = e^{7x^2}$

$g'(x) = e^{7x^2} \cdot (14x)$

$= 14xe^{7x^2}$

④ $h(x) = 2^{3x^4}$

$h'(x) = 2^{3x^4} \cdot \ln(2) \cdot 12x^3$

$= 12x^3\, 2^{3x^4} \ln 2$

⑤ $\frac{d}{dx} \sin^{-1}(7x^4) = \frac{1}{\sqrt{1-(7x^4)^2}} \cdot 28x^3 = \frac{28x^3}{\sqrt{1-(7x^4)^2}}$

⑥ $\frac{d}{dx}(x^2y + 2y = 16) = x^2y + 2y = 16$

$[x^2 \cdot y' + y \cdot 2x] + 2y' = 0$

$x^2y' + 2xy + 2y' = 0$

$x^2y' + 2y' = -2xy$

$y'(x^2+2) = -2xy$

$y' = \frac{-2xy}{x^2+2}$

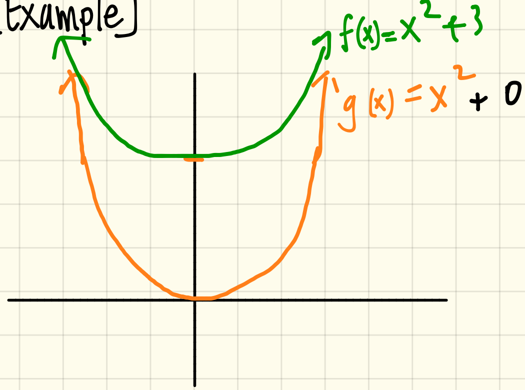
$\boxed{\text{new}}$ Antiderivatives

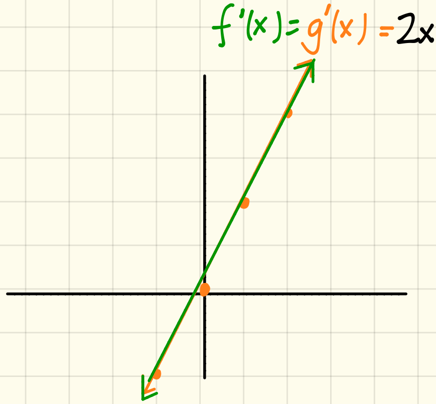
Before
We have been computing derivatives (differentiation) where we start with the position function $f(x)$ & we "do something" to $f(x)$ to come up with the derivatives $f'(x)$.

Now
We are going into the "other direction" where we start with $f(x)$ & we look at another function $F(x)$ such that $f(x)$ is the derivative of $F(x)$ [i.e. $F'(x) = f(x)$].

Let's consider that $f'(x) = g'(x)$ for all x, then $f(x) = g(x) + c$.

[Example]





Conclusion $g'(x) = 2x$ and $f'(x) = 2x$. f and g have the same derivative but differ by a constant.

Definition

If $F(x)$ is a function such that $F'(x) = f(x)$, then $F(x)$ is called the antiderivative of $f(x)$.

Also, if $F(x)$ is an antiderivative $f(x)$, then $F(x) + c$ (for some constant c) is the most general antiderivative of $f(x)$.

[Example] $x^2 + c$ is the most general antiderivative for $2x$.

Question: How can we find antiderivatives of power functions?

We can use what we know about some derivatives.

So here we go:

antiderivatives of $x \Rightarrow \frac{x^2}{2} + c$

$$x^2 \Rightarrow \frac{x^3}{3} + c$$

$$x^3 \Rightarrow \frac{x^4}{4} + c$$

$$\vdots$$

$$\boxed{x^n \Rightarrow \frac{x^{n+1}}{n+1} + c} \longrightarrow \text{power rule for antidifferentiation}$$

[Examples] Find the most general antiderivatives.

① $f(x) = x^5 + 3x^3 + \frac{2}{x^2} - 5e^{-x}$

Rewrite $= x^5 + 3x^3 + 2x^{-2} - 5e^{-x}$

$F(x) = \frac{x^6}{6} + 3\left(\frac{x^4}{4}\right) + 2\left(\frac{x^{-1}}{-1}\right) - -5e^{-x} + C$

$= \frac{1}{6}x^6 + \frac{3}{4}x^4 - \frac{2}{x} + 5e^{-x} + C$

② $g(x) = x^4 + 2$

$G(x) = \frac{x^5}{5} + 2x + C$

③ $h(x) = x^{-2} + \pi$

$H(x) = \frac{x^{-1}}{-1} + \pi x + C$

$= -\frac{1}{x} + \pi x + C$

④ $j(x) = 2x - \frac{3}{x^4}$

Rewrite $= 2x - 3x^{-4}$

$J(x) = 2\left(\frac{x^2}{2}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$

$= x^2 + \frac{1}{x^3} + C$

WARNING: We do not have analogies of product, quotient or chain rules for antidifferentiation... (yet!)

⑤ $f(x) = \frac{x^3 + x^2 + x}{x}$

Rewrite $= \frac{x^3}{x} + \frac{x^2}{x} + \frac{x}{x} = x^2 + x + 1$

$F(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + C$

Remember:

$$\frac{d^2}{dt^2} s(t) = \frac{d}{dt} v(t) = a(t)$$

position function

velocity function

acceleration function

$f(x)$ = position

$f'(x)$ = velocity

$f''(x)$ = acceleration

Integration → the process of antidifferentiation of a function.

Definition

Given a function, F(x), an antiderivative of f(x) is any function F(x) such that $F'(x) = f(x)$.

If F(x) is any antiderivative of f(x), then the most general antiderivative of f(x) is called an indefinite integral and denoted as:

$$\int f(x)\,dx = \underline{F(x) + C}$$

Integral symbol

integrand

"x" is the integration variable

antiderivative

[Examples] Integrate the functions.

① $\int x^4+3x-9\,dx = \frac{x^5}{5} + \frac{3x^2}{2} - 9x + C$

② $\int x^2-2x-5\,dx = \frac{x^3}{3} - \frac{2x^2}{2} - 5x + C$

$= \frac{1}{3}x^3 - x^2 - 5x + C$

③ $\int 2-5x^2\,dx = 2x - \frac{5x^3}{3} + C$

Why is $\int f(x)\,dx = F(x) + c$ called an indefinite integral?

The reason is because we are just finding the antiderivative, not evaluating the antiderivative.

What about other types of functions?

Antiderivatives for Trig Functions:

$$\int \sin x\,dx = -\cos x + C$$

$$\int \csc x \cot x\,dx = -\csc x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \csc^2 x\,dx = -\tan x + c$$

$$\int \tan x\,dx = \ln|\sec x| + c$$

$$\int \cot x = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

Antiderivatives for Inverse Trig Functions:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

Antiderivations for Polynomials:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

Antiderivatives for Exponential Rules/Logarithmic Rules

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

Position Function Problems

Let's say $f'(x) = x^3$ and $f(0) = 4$. What is the position function.

$f'(x) = x^3$

$f(x) = \frac{x^4}{4} + c$

Now we know that $f(0) = 4$, so... $f(0) = \frac{(0)^4}{4} + c = 4$

$c = 4$

The position function is $f(x) = \frac{x^4}{4} + 4$ when $f(0) = 4$.

[Example] Find the position function.

① $f'(t) = 2t + 9t^2$, $f(1) = 2$

$f(t) = \frac{2t^2}{2} + \frac{9t^3}{3} + c$

$= t^2 + 3t^3 + c$

since $f(1) = 2$, $f(1) = (1)^2 + 3(1)^2 + c = 2$

$= 1 + 3 + c = 2$

$= 4 + c = 2$

$c = -2$

Therefore when $f(1) = 2$, $f(t) = t^3 + 3t^2 - 2$.

② $f''(x) = x$, $f(0) = -3$, $f'(0) = 2$

$f''(x) = x$

$f'(x) = \frac{x^2}{2} + c$

$f(x) = \frac{x^3}{6} + cx + d =$

$f'(0) = \frac{0^2}{2} + c = 2$

$c = 2$

$f(0) = \frac{0^3}{6} + 2(0) + d = -3$

$d = -3$

Therefore, $f(x) = \frac{x^3}{3} + 2x - 3$ when $f(0) = -3$ and $f'(0) = 2$.