4.1 Derivatives of Inverse Trigonometric Functions

Standard:

MCD1e

01d Implicit Differentiation & Chain Rule

① $\frac{d}{dx}[2x+\sin(2x^2)+5] = 2+[\cos(2x^2)\cdot 4x]$

$= 2+4x\cos(2x^2)$

② $\frac{d}{dx}(x^4y+4y^3=10x) = [(x^4)(y')+(y)(4x^3)+12y^2\cdot y' = 10$

$x^4y'+4x^3y+12y^2y' = 10$

$x^4y'+12y^2y' = 10-4x^3y$

$y'(x^4+12y^2) = 10-4x^3y$

$y' = \dfrac{10-4x^3y}{x^4+12y^2}$

③ $y = \cos^2(4x+3) = (\cos 4x+3)^2$

$y' = 2(\cos(4x+3))\cdot(-\sin(4x+3))\cdot 4$

$= -8\cos(4x+3)\sin(4x+3)$.

$\boxed{\text{new}}$ Inverse Trigonometric Function Derivatives

- The inverse trig functions are $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$.

$\boxed{\text{please note:}}$ These do not mean $\frac{1}{\sin x}$, $\frac{1}{\cos x}$, $\frac{1}{\tan x}$ $\longrightarrow$ these are reciprocal functions like $\csc x$, $\sec x$ & $\cot x$.

- Inverse Trig Functions are inverses of each trig functions.

(For example) $\sin^{-1}x \rightarrow$ means "inverse sine" where $\sin^{-1}x$ is the angle whose sine is x.

$$\sin\theta = x \qquad\qquad \sin y = x$$
$$\theta = \sin^{-1}x. \qquad\qquad \sin^{-1}x = y.$$

Let's consider $y = \sin^{-1}x$. Derive the derivative.

$$y = \sin^{-1}x$$

So, $\sin y = x$.

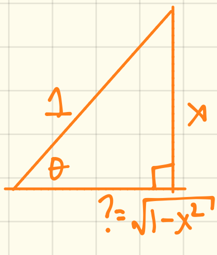
Then if we find the derivative:

$$\cos y \cdot y' = 1$$
$$y' = \frac{1}{\cos y}.$$

Since $y = \sin^{-1}x$, then $y' = \dfrac{1}{\cos(\sin^{-1}x)}$.

Let $\theta = \sin^{-1}x$

Then $\sin\theta = x$ $\therefore$ $\sin\theta = \frac{x}{1}$



$$x^2 + ?^2 = 1^2$$
$$?^2 = 1 - x^2$$
$$? = \sqrt{1-x^2}$$

Now, $y' = \dfrac{1}{\cos\theta}$ & $\cos\theta = \dfrac{\sqrt{1-x^2}}{1}$,

Therefore $y' = \dfrac{1}{\sqrt{1-x^2}}$. $\blacksquare$

Let's consider $y = \tan^{-1}x$. Derive the derivative.

$$y = \tan^{-1}x$$

So, $\tan y = x$.

Now, take the derivative:

$$\sec^2 y \cdot y' = 1$$

$$y' = \frac{1}{\sec^2 y}$$

Since $y = \tan^{-1}x$, $y' = \dfrac{1}{\sec^2(\tan^{-1}x)}$.

Let $\theta = \tan^{-1}x$.
Then $\tan\theta = x$

Background Information $\rightarrow$ Trig Identities
$\sin^2 x + \cos^2 x = 1$
$1 + \tan^2 x = \sec^2 x$.

So since $1 + \tan^2 x = \sec^2 x$,

$$y' = \frac{1}{1+\tan^2\theta}$$

Furthermore, since $\tan^2\theta$,

$$y' = \frac{1}{1+x^2}.$$

$\blacksquare$

Derivatives of Inverse Trig Functions

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$\boxed{\text{note:}}$ Chain Rule on the "x".

[Examples]

① $f(x) = \sin^{-1}(x^2)$

$f'(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$

② $\frac{d}{dx}\arcsin(3x^2) = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot 9x$

$= \frac{9x}{\sqrt{1-9x^4}}$

③ $\frac{d}{dx}\arccos\left(\frac{1}{x}\right) = \frac{-1}{\sqrt{1-(x^{-1})^2}} \cdot (-x^{-2})$

$= \frac{1}{x^2\sqrt{1-x^{-2}}}$

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- 1-7, 13, 21, 27,
- Also derive the derivative of $y = \cos^{-1}x$.