4.2 Area Between Curves

Riemann Sums

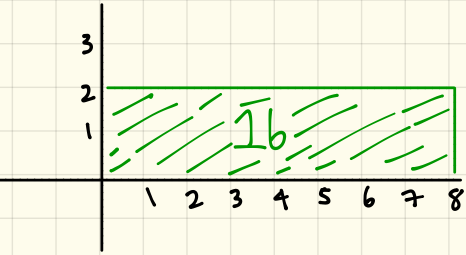
Standards:

MC11

MC11b

$\boxed{\text{Old}}$ Area of Rectangles

Let's consider the following situation:
Suppose you travelled at $2\,\frac{\text{ft}}{\text{sec}}$ for 8 seconds. How far did you go?



Answer: 16 feet.

Why?

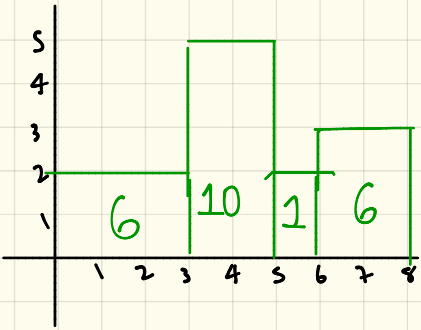
The rate is $2\frac{\text{ft}}{\text{sec}}$ and 8 seconds of travel time.

$$\text{Distance} = \text{Rate} \cdot \text{Time}$$
$$= 2\frac{\text{ft}}{\text{sec}} \quad 8\text{ secs}$$
$$= 16 \text{ feet}.$$

Conclusion

The area of rectangles gives the total distance travelled.

Let's consider a slightly realistic scenario:



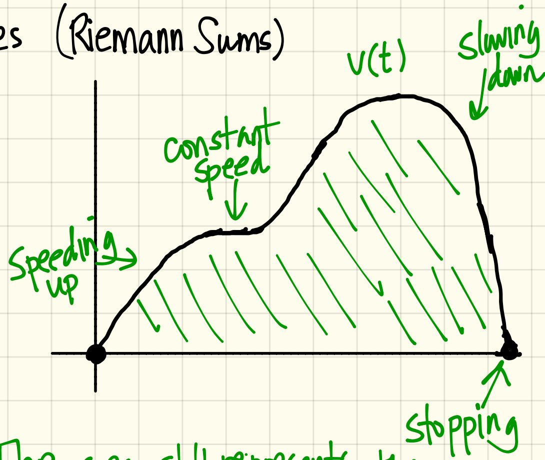
So the total distance travelled:

$$6 + 10 + 1 + 6 = 23 \text{ feet}.$$

NEW Area & Distances under curves (Riemann Sums)

What would be a more realistic situation?

A velocity function that varies continuously with time.



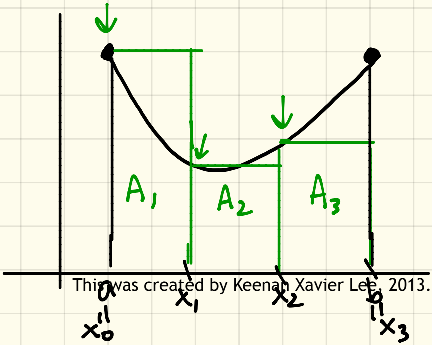
Dilemma: But how do we find the area under this curve?

The area still represents the total distance travelled.

Basic Idea: We need to understand how to compute area of curved regions. We do this by "approximating" the region with rectangles and adding their areas (Riemann sums).

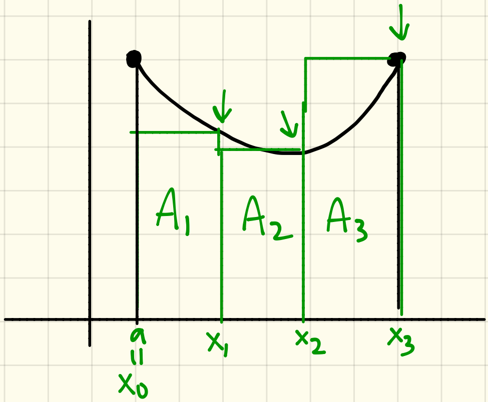
You can use Right Endpoint Approximation, Left Endpoint Approximation, and Midpoint Approximation.

Left Rectangular Approximation Method (LRAM)



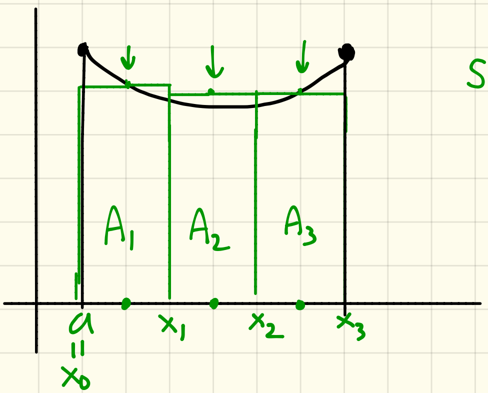
$$\text{Sum} = A_1 + A_2 + A_3$$
$$= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2)$$
$$= \Delta x\left[f(x_0) + f(x_1) + f(x_2)\right]$$

Right Rectangular Approximation Method (RRAM)



$$\begin{aligned}\text{sum} &= A_1 + A_2 + A_3 \\ &= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) \\ &\quad + f(x_3)(x_3 - x_2) \\ &= \Delta x\left[f(x_1) + f(x_2) + f(x_3)\right]\end{aligned}$$

Midpoint Rectangular Approximation Method (MRAM)



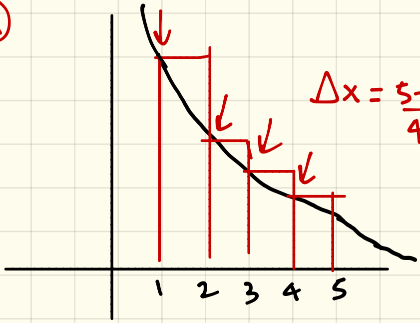
$$\begin{aligned}\text{sum} &= A_1 + A_2 + A_3 \\ &= f\left(\frac{x_0 + x_1}{2}\right)(x_0 - x_1) + f\left(\frac{x_2 + x_1}{2}\right)(x_2 - x_1) \\ &\quad + f\left(\frac{x_3 + x_2}{2}\right)(x_3 - x_2) \\ &= \Delta x\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_2 + x_1}{2}\right) + f\left(\frac{x_3 + x_2}{2}\right)\right].\end{aligned}$$

$$\Delta x = \frac{b - a}{n}$$

a = start
b = stop
n = # of rectangles

[Example] Find the area using $f(x)=\frac{1}{x}$ from $x=1$ to $x=5$ using 4 rectangles using ⓐ RRAM, ⓑ LRAM and ⓒ MRAM.

ⓐ



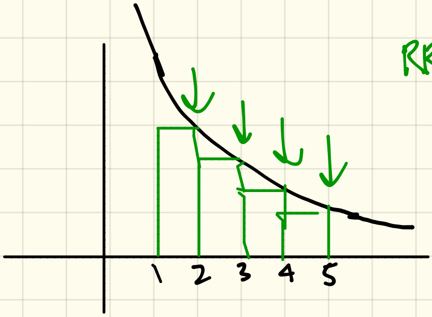
$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1$

LRAM sum $= \Delta x \left[f(1)+f(2)+f(3)+f(4)\right]$

$= 1\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$

≈ 2.083

ⓑ

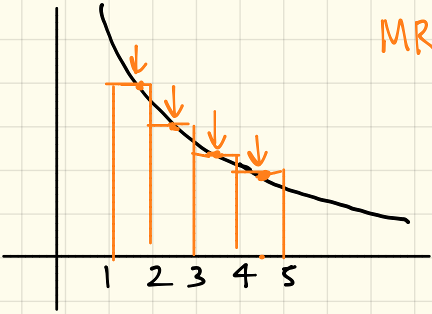


RRAM sum $= \Delta x \left[f(x_2)+f(x_3)+f(x_4)+f(x_4)\right]$

$= 1\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)$

≈ 1.283

ⓒ



MRAM sum $= \Delta x \left[f(1.5)+f(2.5)+f(3.5)+f(4.5)\right]$

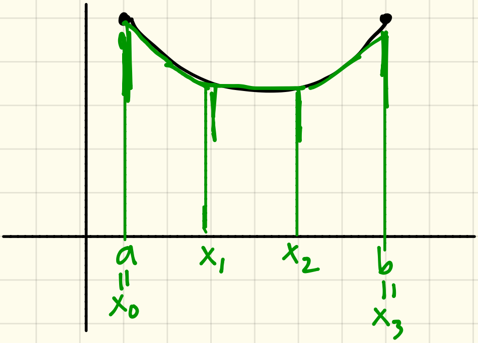
$= 1\left[\frac{1}{1.5}+\frac{1}{2.5}+\frac{1}{3.5}+\frac{1}{4.5}\right]$

≈ 1.575

LRAM is an overapproximation & RRAM is an underapproximation.

$\boxed{\text{Note:}}$ The more rectangles applied, the more areas being computed, and the accurate the approximation of the area under the curve will be.

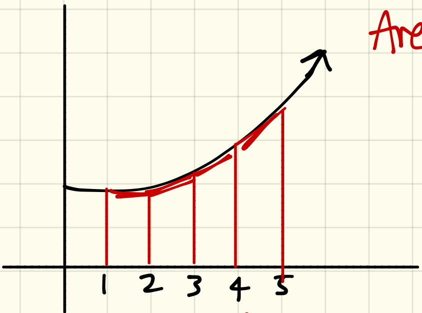
Trapezoidal Rule for Approximation



Area =

$$\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

[Example] Approximate using Trapezoidal Rule for $f(x) = 1 + x^2$ using $n = 4$ (4 rectangles) from $x = 1$ to $x = 5$



$$\text{Area} = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2(x_3) + 2(x_4) + f(x_5)\right]$$

$$= \frac{1}{2}\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right]$$

$$= \frac{1}{2}\left[2 + 2(5) + 2(10) + 2(17) + 26\right]$$

$$= 46$$

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = \frac{4}{4}$$