4.4 Mean Value Theorem

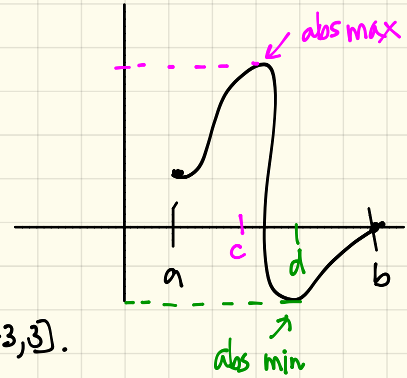
Standards:

MCD2

MCD2c

01d Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$, then $f(x)$ attains its abs max & min somewhere on $[a,b]$.



[Example] Find abs max/min for $f(x) = x^5 - 80x$ on $[-3,3]$.

Step 1: $f(x) = x^4 - 80x$

$f'(x) = 5x^4 - 80$

$$5x^4 - 80 = 0$$
$$5(x^4 - 16) = 0$$
$$x^4 - 16 = 0$$
$$x^4 = 16$$
$$x = 2, -2.$$

Step 2/3:

critical #'s	endpoints
$f(2) = -128$	$f(3) = 3$
$f(-2) = 128$	$f(-3) = -3$

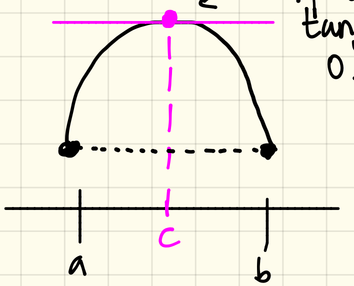
Step 4: The abs max is 128 occuring at $x=2$ & abs min is -128 occuring at $x=2$.

$\boxed{\text{new}}$ Mean Value Theorem

please note: The main result of this topic is the Mean Value Theorem. To prove & understand this concept, we need to know the Rolle's Theorem.

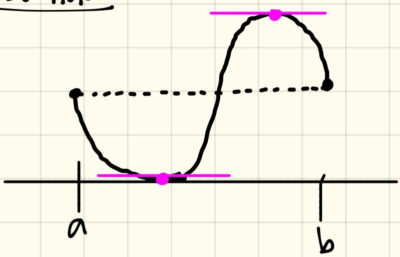
Basic Idea of Rolle's Theorem

slope of the tangent line is 0.



If the endpoint values of f are equal, then f must have an extreme value at some number strictly between the endpoints, provided that f is continuous on the interval & f is differentiable, then $f'(x) = 0$.

Also note:



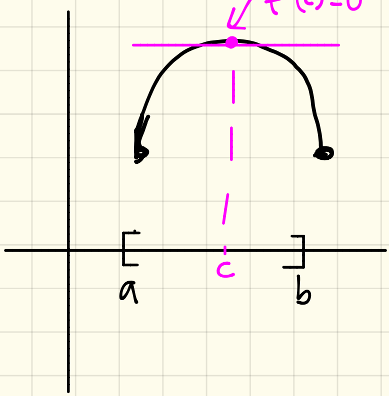
A slope might be 0 twice (or more) between numbers where the values of f are equal.

Rolle's Theorem

So suppose that

1. $f(x)$ is continuous on $[a,b]$
2. $f(x)$ is differentiable on (a,b)
3. $f(a) = f(b)$

Then, there must exist at least a number c in $[a,b]$ such that $f'(c) = 0$.



Rolle's Theorem will lead us to the Mean Value Theorem.

Mean Value Theorem

So suppose that
1. $f(x)$ is continuous on $[a,b]$
2. $f(x)$ is differentiable on (a,b)

Then, there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Note: The Mean Value Theorem is a more general case of Rolle's Theorem which has additional hypothesis that $f(a)=f(b)$ and in conclusion states that there is a number c such that $f'(c)=0$.

[Example 1] Find a point c satisfying the conclusion of MVT for $f(x)=\frac{1}{x}$ in $[2,8]$.

First: Is f continuous on $[2,8]$? yes

Second: Is f differentiable on $(2,8)$? yes

Now, $f'(c) = \frac{f(b)-f(a)}{b-a}$

Let $a=2$ & $b=8$. So, $\frac{f(8)-f(2)}{8-2} = \frac{-1}{16}$

Also, $f(c) = \frac{1}{c} \overset{\text{Rewrite}}{=} c^{-1}$. $f'(x) = -1c^{-2} = \frac{-1}{c^2}$.

So now set $f'(c) = \frac{-1}{16} \longrightarrow \frac{-1}{c^2} = \frac{-1}{16}$

$-c^2 = -16$

$c^2 = 16$

$c = \pm 4.$

But since $c=-4$ isn't in $[2,8]$, $c=-4$ is not the answer.

Therefore $c=4$.

[Example 2] Find a point c satisfying the conclusion of MVT for $f(x)=\sqrt{x}$ on $[9,25]$.

First: Is f continuous on $[9,25]$? yes

Second: Is f is differentiable on $(9,25)$? yes

Now, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Let $a=9$, $b=25$. So, $\dfrac{f(25)-f(9)}{25-9} = \dfrac{5-3}{25-9} = \dfrac{1}{8}$

Also $f(c)=\sqrt{c}$. Now, $f'(c) = \dfrac{1}{2}c^{-1/2} = \dfrac{1}{2\sqrt{c}}$

So, $\dfrac{1}{2\sqrt{c}} = \dfrac{1}{8}$

$8 = 2\sqrt{c}$

$4 = \sqrt{c}$

$(4)^2 = (\sqrt{c})^2$

$16 = c$

Therefore, $c=16$ meeting MVT's requirements.