

4.5 Substitution Rule

Using Trigonometric Identities

Standards:

MC11

MC11d



[Old] Substitution Rule for Integration

Recall: using u -substitution technique to compute integrals involving compositions of functions.

Example Integrate.

$$\int x^2 \sqrt{5+2x^3} dx = \frac{1}{6} \int \sqrt{u} du = \int \frac{1}{6} (u)^{1/2} du = \frac{1}{6} \left[\frac{u^{3/2}}{3/2} \right] + C$$

$$u = 5+2x^3 \\ du = 6x^2 \Rightarrow \frac{1}{6} du = x^2 dx$$

$$= \frac{1}{6} \left[\frac{2}{3} u^{3/2} \right] + C = \frac{1}{9} (5+2x^3)^{3/2} + C = \frac{1}{9} \sqrt{(5+2x^3)^3}$$

[New] Trig Substitution

With integrating trig functions, the basic idea is to :

- ① rewriting integral functions using trig identities.
- ② use u -substitution technique

Case 1 - using reciprocal formulas

$$\sin x = \frac{1}{\csc x} \quad \cos x = \frac{1}{\sec x} \quad \tan x = \frac{1}{\cot x} \quad \csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$$

[Example] Integrate.

$$\textcircled{1} \int \frac{dx}{\cos^2 x} = \int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C.$$

$$\textcircled{2} \int \cot 7x \, dx = \int \frac{\cos 7x}{\sin 7x} \, dx = \frac{1}{7} \int \frac{1}{u} \, du = \frac{1}{7} \int u^{-1} \, du =$$

$$u = \sin 7x$$

$$du = \cos 7x \cdot 7 \, dx$$

$$du = 7 \cos 7x \, dx \Rightarrow \frac{1}{7} du = \cos 7x$$

$$= \frac{1}{7} [\ln |u|] + C = \frac{1}{7} \ln |\sin 7x| + C.$$

Case 2: $\int \cos^k x \sin^l x \, dx$

when k & l are positive integers & at least one of k or l is odd.

* useful technique: save the odd factor & use $\sin^2 x + \cos^2 x = 1$.

[Examples.]

$$\textcircled{1} \int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

$$u = \cos x$$

$$du = -\sin x \, dx \Rightarrow -du = \sin x \, dx$$

$$= - \int (1 - u^2) u^2 \, du = - \int u^2 - u^4 \, du = - \left[\frac{u^3}{3} - \frac{u^5}{5} + C \right] = -\frac{u^3}{3} + \frac{u^5}{5} - C$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} - C.$$

$$\textcircled{2} \int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

$$u = \sin x$$

$$du = \cos x \, dx$$

$$= \int 1 - u^2 \, du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

Case 3: $\int \cos^k x \sin^l x dx$

when k & l are positive integers & are both even.

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x)) \quad \sin(2x) = 2 \sin x \cos x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

[Examples]

$$\textcircled{1} \int \sin^2 x \cos^2 x dx = \int \frac{1}{2}(1 - \cos(2x)) \cdot \frac{1}{2}(1 + \cos(2x)) dx$$

$$= \frac{1}{4} \int (1 - \cos(2x))(1 + \cos(2x)) dx = \frac{1}{4} \int 1 - \cos^2 2x dx$$

$$= \frac{1}{4} \int 1 - \cos^2 2x dx = \frac{1}{4} \int 1 - \left[\frac{1}{2}(1 + \cos 4x) \right] dx$$

$$= \frac{1}{4} \int 1 - \frac{1}{2} - \frac{1}{2} \cos(4x) dx = \frac{1}{4} \int \frac{1}{2} - \frac{1}{2} \cos 4x dx$$

$$= \frac{1}{4} \int \frac{1}{2} dx - \frac{1}{4} \cdot \frac{1}{2} \int \cos 4x dx = \frac{1}{4} \int \frac{1}{2} dx - \frac{1}{32} \int \cos u dx$$

$$u = 4x \\ du = 4 dx \Rightarrow \frac{1}{4} du = dx$$

$$= \frac{1}{4} \left[\frac{1}{2} x \right] - \frac{1}{32} [\sin u] + C = \frac{1}{8} x - \frac{1}{32} \sin 4x + C.$$

Case 4: $\int \tan^k x \sec^l x dx$

when k & l are positive integers & k is even.

Save $\sec^2 x$ factor, use $\sec^2 x = 1 + \tan^2 x$ & $u = \tan x$

[Examples]

① $\int \tan^2 x \sec^4 x dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x dx$

$$u = \tan x \\ du = \sec^2 x dx$$

$$= \int u^2 (1 + u^2) du = \int u^2 + u^4 du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

② $\int \sec^4 x dx = \int \sec^2 x \sec^2 x dx = \int (1 + \tan^2 x) (\sec^2 x) dx$

$$= \int \sec^2 x + \tan^2 x \sec^2 x dx = \int \sec^2 x + \int \tan^2 x \sec^2 x$$

$$u = \tan x \\ du = \sec^2 x$$

$$= \int \sec^2 x + \int u^2 du = \tan x + \frac{u^3}{3} + C$$

$$= \tan x + \frac{\tan^3 x}{3} + C$$

Case 5: $\int \tan^k x \sec^l x dx$

when k & l are positive integers & l is odd

save $\tan x \sec x$ factor, use $\tan^2 x = \sec^2 x - 1$ & $u = \sec x$

[Examples]

① $\int \tan^5 x \sec^3 x dx = \int \tan^4 x \sec^2 x \tan x \sec x dx$

$= \int (\sec^2 x - 1)^2 (\sec^2 x) \tan x \sec x dx = \int (u^2 - 1)^2 u^2 du$

$u = \sec x$

$du = \sec x \tan x dx$

$= \int (u^4 - 2u^2 + 1) u^2 du = \int u^6 - 2u^4 + u^2 du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$

$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$