4.6 Integration By Parts

Standards:

MC11

MC11d

$\boxed{01d}$ Substitution Rule for Integration

① $\int x^3 \cos(x^4+2)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}[-\sin u] + c$

$u = x^4+2$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$= -\frac{1}{4}\sin u + c = -\frac{1}{4}\sin(x^4+2) + c.$

② $\int \sin^5 x \cos^2 x\,dx = \int \sin^4 x \cos^2 x \sin x\,dx = \int (\sin^2 x)^2 \cos^2 x \sin x\,dx$

$= \int (1-\cos^2 x)^2 \cos^2 x \sin x\,dx = -\int (1-u^2)^2 u^2\,du = \int (1-u^2)(1-u^2)u^2\,du$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$= \int (1-2u^2+u^4)u^2\,du = \int u^2 - 2u^4 + u^6\,du = \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + c$

$= \frac{\cos^3 x}{3} - \frac{2\cos^7 x}{7} + \frac{\cos^7 x}{7} + C$

$\boxed{\text{new}}$ Integration By Parts

Let's consider the product rule for differentiation:

$$\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Now let's integrate both sides:

$$\int \frac{d}{dx}[f(x)\cdot g(x)]\,dx = \int f'(x)g(x) + f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x) + f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Now let:

$u = f(x) \quad du = f'(x)\,dx$

$v = g(x) \quad dv = g'(x)\,dx$

$$uv = \int v\,du + \int u\,dv$$

Finally,

$$\boxed{uv - \int v\,du = \int u\,dv} \leftarrow \text{Integration by Parts Formula.}$$

$\int u\,dv = uv - \int v\,du$

So, to evaluate $\int u\,dv$, we need to:

1. identify the pieces u & dv
2. calculate du & v
3. substitute the pieces in the correct place, and integrate.

[Example 1] Evaluate $\int x e^{2x}\,dx$

$$\int x e^{2x}\,dx = (x)\left(\tfrac{1}{2}e^{2x}\right) - \int \tfrac{1}{2}e^{2x}\,dx$$

$u = x$ $\quad dv = e^{2x}dx$

$du = 1dx$ $\quad v = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right] + c$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.$$

[Example 2] Evaluate $\int \sqrt{t}\,\ln t\,dt$.

$$\int \sqrt{t}\,\ln t\,dt = (\ln t)\left(\tfrac{2}{3}t^{3/2}\right) - \int \left(\tfrac{2}{3}t^{3/2}\right)\left(\tfrac{1}{t}\right)dt$$

$u = \ln t$ $\quad dv = t^{1/2}$

$du = \frac{1}{t}dt$ $\quad v = \frac{t^{3/2}}{3/2} = \frac{2}{3}t^{3/2}$

$$= \frac{2\sqrt{t^3}\ln t}{3} - \frac{2}{3}\int \frac{t^{3/2}}{t}dt = \frac{2\sqrt{t^3}\ln t}{3} - \frac{2}{3}\int t^{1/2}dt$$

$$= \frac{2\sqrt{t^3}\ln t}{3} - \frac{2}{3}\left[\frac{2}{3}t^{3/2}\right] + c = \frac{2\sqrt{t^3}\ln t}{3} - \frac{4\sqrt{t^3}}{9} + c$$

please note: $\ln x$ is NEVER a good choice for dv.

Guidelines for choosing "u":

L - Logarithmic
I - Inverse Functions
A - Algebraic (polynomials)
T - Trigonometric Functions
E - Exponential Functions

[Example 3] Evaluate $\int t^2 e^t \, dt$

$$\int t^2 e^t \, dt = t^2 e^t - \int 2te^t \, dt = t^2 e^t - \left[2te^t - \int 2e^t \, dt\right]$$

$u = t^2$ $\quad dv = e^t \, dt$

$du = 2t\,dt$ $\quad v = e^t$

$u = 2t$ $\quad dv = e^t \, dt$

$du = 2dt$ $\quad v = e^t$

$$= t^2 e^t - \left[2te^t - 2\int e^t \, dt\right] = t^2 e^t - \left[2te^t - 2e^t\right] + c$$

$$= t^2 e^t - 2te^t + 2e^t + c.$$

$\boxed{\text{note:}}$ sometimes the integrand may occur again in the process.

[Example 4] $\int e^x \cos x \, dx$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

$u = \sin x$ $dv = e^x dx$

$du = \cos x dx$ $v = e^x$

$u = \cos x$ $dv = e^x dx$

$du = -\sin x dx$ $v = e^x$

$$= e^x \cos x + e^x \sin x - \underline{\int e^x \cos x \, dx}$$

*reoccurance... so you got to solve.

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$+\int e^x \cos x \, dx = \qquad + \int e^x \cos dx$$

$$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\frac{2\int e^x \cos x \, dx}{2} = \frac{e^x \cos x + e^x \sin x}{2}$$

$$\int e^x \cos x \, dx = \frac{e^x \cos x + e^x \sin x}{2} + C$$