4.7 Partial Fraction Integration

$\boxed{\text{Old-A}}$ Integration Techniques

- Power Rule → sum of functions
- Substitution Rule → composition of functions
- Integration by Parts → product of functions

[Examples] Determine which technique is appropriate. (Do not work out.)

① $\int xe^x dx$

by parts

② $\int xe^{2x}$

by parts

③ $\int xe^{x^2} dx$

u-substitution

④ $\int x^2 e^{x^3} dx$

u-substitution

⑤ $\int x^2 e^x dx$

by parts (twice)

④ $\int x^2 e^{2x} dx$

by parts (twice)

⑦ $\int x^2 e^{x^2} dx$

can't do

⑧ $\int x^3 e^{x^2} dx$

by parts

⑨ $\int x^2 + 5\, dx$

power rule

more old... Long Polynomial Division with improper fractions

④ $\dfrac{4x^2+4x-3}{2x-1} = 2x+3$

$$\begin{array}{r} 2x+3 \\ 2x-1 \overline{) 4x^2+4x-3} \\ \ominus 4x^2 \oplus 2x \\ \hline 6x-3 \\ \oplus 6x \oplus 3 \\ \hline 0 \end{array}$$

⑤ $\dfrac{x^2+5x+3}{x+6} = x-1+\dfrac{9}{x+6}$

$$\begin{array}{r} x-1 \\ x+6 \overline{) x^2+5x+3} \\ \ominus x^2 \ominus 6x \\ \hline -1x+3 \\ \oplus 1x \oplus 6 \\ \hline 9 \end{array}$$

$\boxed{\text{new-A}}$ Partial Fractions Integration

The method of partial fractions handles any rational function, $\frac{p(x)}{q(x)}$, with $p(x)$ & $q(x)$ are polynomials (quotients).

Case 1: Improper Fractions $\longrightarrow$ Long Polynomial Division

* improper fractions — degree is higher in numerator than denominator.

1. $\displaystyle\int \frac{12x^3-11x^2+9x+18}{4x+3}\,dx$

$$
\begin{array}{r}
3x^2-5x+6 \\
4x+3\,\overline{)\,12x^3-11x^2+9x+18} \\
-12x^3-9x^2 \\
\hline
-20x^2+9x^2 \\
+20x^2+15x^2 \\
\hline
24x^2+18 \\
-24x^2-18 \\
\hline
0.
\end{array}
$$

$$= \int 3x^2-5x+6\,dx = \frac{3x^3}{3} - \frac{5x^2}{2} + 6x + C = x^3 - \frac{5}{2}x^2 + 6x + C.$$

② $\int \frac{x^3}{x+1}\,dx$

$$\begin{array}{r} x^2 - x + 1 \\ x+1 \,\big)\, x^3 + 0x^2 + 0x + 0 \\ \ominus x^3 \, \cancel{+}\ominus\, x^2 \\ \hline -1x^2 + 0x \\ \oplus\, x^2 \oplus 1x \\ \hline 1x + 0 \\ \ominus\, x \,\cancel{+}\ominus\, 1 \\ \hline -1 \end{array}$$

$$= \int x^2 - x + 1 - \frac{1}{x+1}\,dx = \int x^2 - x + 1\,dx - \int \frac{1}{x+1}\,dx$$

$u = x+1$
$du = 1\,dx$ $\ldots \int \frac{1}{u}\,du = \ln|u|$
$= \ln|x+1|$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C.$$

③ $\int \frac{2x^3+4x^2-5}{x+3}\,dx$

$$\begin{array}{r} 2x^2 - 2x + 6 \\ x+3 \,\big)\, 2x^3 + 4x^2 + 0x - 5 \\ \ominus 2x^3 \,\cancel{+}\ominus\, 6x^2 \\ \hline -2x^2 + 0x \\ \oplus 2x^2 \oplus 6x \\ \hline 6x - 5 \\ \ominus 6x \,\cancel{+}\ominus\, 18 \\ \hline -23 \end{array}$$

$$= \int 2x^2 - 2x + 6 - \frac{23}{x+3}\,dx = \int 2x^2 - 2x + 6\,dx - \int \frac{23}{x+3}\,dx$$

$u = x+3$
$du = 1\,dx$
$\frac{1}{23}du = 23\,dx$ $\ldots 23\int \frac{1}{u} =$
$= 23\ln|u|$
$= 23\ln|x+3|$

$$= \frac{2x^3}{3} - \frac{2x^2}{2} + 6x - 23\ln|x+3| + C.$$

more old... Adding & Subtracting Rational Expressions

[Examples]

1. $$\frac{x-1}{x} + \frac{x+2}{x^2} = \frac{x-1(x)}{x(x)} + \frac{x+2}{x^2} = \frac{x^2-x}{x^2} + \frac{x+2}{x^2} = \frac{x^2-x+x+2}{x^2}$$

$$= \frac{x^2+2}{x^2}.$$

2. $$\frac{x-1}{x^2+3x+2} + \frac{x}{x+1} = \frac{x-1}{(x+2)(x+1)} + \frac{x}{x+1} = \frac{x-1}{(x+2)(x+1)} + \frac{x(x+2)}{(x+1)(x+2)}$$

$$= \frac{x-1+x(x+2)}{(x+1)(x+2)} = \frac{x-1+x^2+2x}{(x+1)(x+2)} = \frac{x^2+3x-1}{(x+1)(x+2)}$$

3. $$\frac{5}{x+3} - \frac{2}{x-1} = \frac{5(x-1)}{x+3(x-1)} - \frac{2(x+3)}{x-1(x+3)} = \frac{5x-5-(2x+6)}{(x+3)(x-1)}$$

$$= \frac{5x-5-2x-6}{(x+3)(x-1)} = \frac{3x-11}{(x+3)(x-1)}$$

$\boxed{\text{new}}$ Partial Fractions Integration

Case 2: proper fractions → partial fractions decompositions

goal: We need to go in reverse from adding & subtracting rational expressions.

Let's consider the rational function: $\frac{3x-11}{x^2+2x-3}$ Deconstruct the rational.

$$\frac{3x-11}{x^2+2x-3}$$

$$= \frac{3x-11}{(x+3)(x-1)}$$ — factor the denominator if necessary

$$= \frac{A}{x+3} + \frac{B}{x-1}$$ — break up factor products into sum

$$= \frac{A(x-1) + B(x+3)}{(x+3)(x-1)}$$ — get common denominator

$$\rightarrow Ax - A + Bx + 3B$$

— group & match variables

$$x: \quad 3 = A + B$$
$$\text{constant}: -11 = -A + 3B$$

— solve system of equations
i.e. elimination is useful usually

$$-8 = 4B$$
$$-2 = B.$$

Find A: $3 = A + (-2)$
$5 = A.$

Ans: $\frac{5}{x+3} + \frac{-2}{x-1}$ ← much easier to integrate!

Let's use this technique to integrate proper fractions!

[Examples] Integrate.

① $\int \frac{3x+5}{(x-3)(2x+1)}dx = \int \frac{A}{x-3} + \frac{B}{2x+1}\, dx$

$\Rightarrow \frac{3x+5}{(x-3)(2x+1)} = \frac{A(2x+1)+B(x-3)}{(x-3)(2x+1)} \rightarrow 2Ax + A + Bx - 3B$

x: $3 = 2A + B$

constant: $5 = A - 3B$

$\Rightarrow 3\begin{bmatrix} 3 = 2A+B \end{bmatrix}$ $5 = A - 3B$

$9 = 6A + 3B$

$5 = A - 3B$

$14 = 7A$

$2 = A.$

Find B: $9 = 6(2) + 3B$

$9 = 12 + 3B$

$-3 = 3B$

$-1 = B$

$= \int \frac{2}{x-3} - \frac{1}{2x+1}dx = \int \frac{2}{x-3}dx - \int \frac{1}{2x+1}dx$

$u = x-3$ $\quad$ $v = 2x+1$

$du = 1dx$ $\quad$ $dv = 2dx$

$2du = 2dx$ $\quad$ $\frac{1}{2}dv = 1dx$

$= 2\int \frac{1}{u}du - \frac{1}{2}\int \frac{1}{v}dv = 2\ln|u| - \frac{1}{2}\ln|v| = 2\ln|x-3| - \frac{1}{2}\ln|2x+1| + C$

(2) $\displaystyle\int \frac{x^2+x+2}{x^2(x-2)}\,dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}\,dx$

$\Rightarrow \quad x^2+x+2 = Ax(x-2) + B(x-2) + Cx^2$

$x^2+x+2 = Ax^2-2Ax+Bx-2B+Cx^2$

x^2: $1 = A+C$

x: $1 = -2A+B$

constant: $2 = -2B$

Find B:

$2 = -2B$

$-1 = B$

Find A:

$1 = -2A + (-1)$

$2 = -2A$

$-1 = A$

Find C:

$1 = (-1) + C$

$2 = C.$

$= \displaystyle\int \frac{-1}{x} + \frac{-1}{x^2} + \frac{2}{x-2}\,dx = \int \frac{-1}{x} - x^{-2}\,dx + \int \frac{2}{x-2}\,dx$

$= -\ln|x| - \dfrac{x^{-1}}{-1} + 2\ln|x-2| + C$

$= \ln|x| + \dfrac{1}{x} + 2\ln|x-2| + C.$