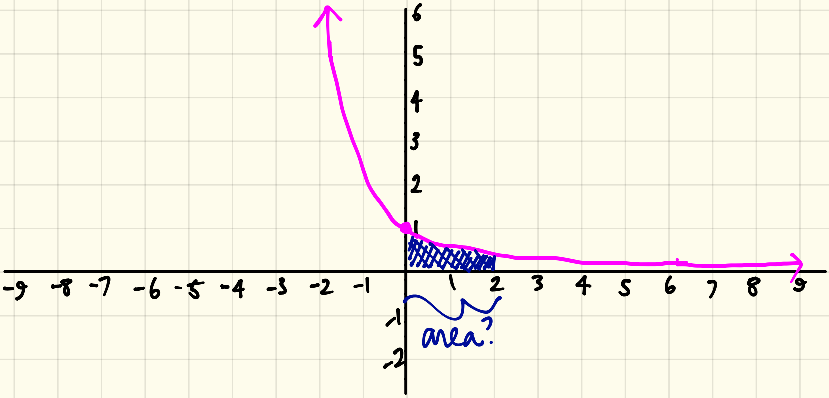
4.8 Improper Integrals

$\boxed{\text{Old}}$ Area – Integration

Let's consider the function: $f(x) = e^{-x}$. Graph the function and find the area between $x=0$ and $x=2$.



Need to integrate:

$$\int_0^2 e^{-x}\,dx = -\int_0^2 e^u\,du = -e^u\Big]_0^2 = -e^{-x}\Big]_0^2 = \left[-e^{-(2)}\right] - \left[-e^0\right] = -\frac{1}{e^2} + 1.$$

$u = -x$

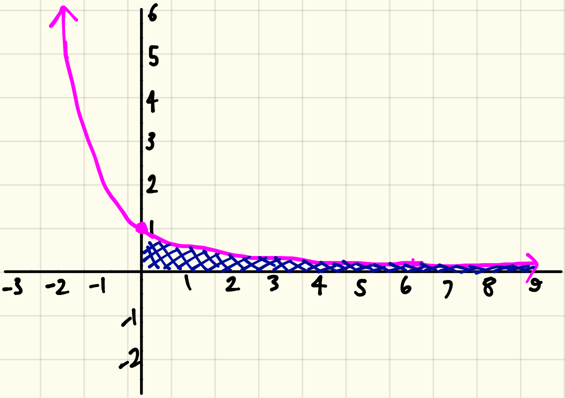
$du = -1\,dx$

$-du = 1\,dx$

We were able to use integration to be able to find the area between 2 boundaries.

$\boxed{\text{new}}$ Improper Integrals

Let's consider the past graph of the function:



Before: finding area between 2 boundaries

Now: finding area when one sided is unbounded or both sides are unbounded.

Case 1: Infinite Integrals, where an infinity is either at one or both limits of integration

$\boxed{1}$ If $f(x)$ is continuous on $[a, \infty)$, then $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{\to\infty} \int_a^{t} f(x)\,dx$

$\boxed{2}$ If $f(x)$ is continuous on $(-\infty, a]$, then $\displaystyle\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty} \int_t^{a} f(x)\,dx$

$\boxed{3}$ If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

(where c is some real number)

This was created by Keenan Xavier Lee, 2013. See my website for more information, lee-apcalculus.weebly.com.

Solutions can be either:

$\boxed{a}$ convergent — the limit exist & is a real number.

$\boxed{b}$ divergent — the limit fails to exist & is infinite ($-\infty$ or ∞).

Suppose we want to find the area starting from 0 going towards positive infinity (∞). How do we find the area?

$$\int_0^{\infty} e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{u}\,du = \lim_{t\to\infty} -e^{u}\Big]_0^t = \lim_{t\to\infty} -e^{-x}\Big]_0^t$$

$u=-x$

$du=-1\,dx$

$-du = dx$

$$= \lim_{t\to\infty}\left[-e^{-t} - -e^{0}\right] = \lim_{t\to\infty} \underbrace{-e^{-t}}_{\to 0} + 1 = 0+1 = 1, \text{ convergent.}$$

[Examples] Integrate.

① $$\int_1^{\infty}\frac{dx}{x^3} = \lim_{t\to\infty}\int_1^t \frac{1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t x^{-3}\,dx = \lim_{t\to\infty}\frac{x^{-2}}{-2}\Big]_1^t$$

$$= \lim_{t\to\infty}\frac{-1}{2x^2}\Big]_1^t = \lim_{t\to\infty}\left[\frac{-1}{2t^2} - \frac{-1}{2(1)}\right] = \lim_{t\to\infty}\left[\underbrace{\frac{-1}{2t^2}}_{\to 0} + \frac{1}{2}\right]$$

$$= 0 + \frac{1}{2} = \frac{1}{2}, \text{ convergent.}$$

② $\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\Big]_1^t = \lim_{t\to\infty}\left[\ln|t| - \ln|1|\right]$

$= \lim_{t\to\infty}\left[\underbrace{\ln|t|}_{\infty} - 0\right] = \infty - 0 = \infty$; divergent.

③ $\int_0^\infty \frac{2\,dx}{x^2+4x+3} = \lim_{t\to\infty}\int_0^t \frac{2}{(x+3)(x+1)}dx$

$\Rightarrow \frac{2}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$

$2 = A(x+1) + B(x+3)$
$2 = Ax + A + Bx + 3B$

x: $0 = A + B$ $\Rightarrow$ $0 = A + B$
constant: $2 = A + 3B$ $\quad -2 = -A - 3B$

$-2 = -2B$ $\quad$ Find A: $0 = A + (1)$
$1 = B$ $\quad -1 = A$

$\lim_{t\to\infty}\int_0^t \frac{-1}{x+3} + \frac{1}{x+1}\,dx = \lim_{t\to\infty} -\ln|x+3| + \ln|x+1|\Big]_0^t =$

$= \lim_{t\to\infty} -\ln|t+3| + \ln|t+1| + \left[+\ln|3| - \ln|1|\right] = \lim_{t\to\infty} \ln\left|\frac{t+1}{t+3}\right| + \ln 3$

$= \lim_{t\to\infty} \ln\left|\frac{1+\frac{1}{t}\to 0}{1+\frac{3}{t}\to 0}\right| + \ln 3 = \ln|1| + \ln|3| = \ln|3|$, convergent.

4 $\displaystyle\int_{-\infty}^{\infty} e^x\,dx = \overset{a}{\lim_{t\to-\infty}} \int_t^0 e^x\,dx + \overset{b}{\lim_{t\to\infty}} \int_0^t e^x\,dx$

evaluate a $\displaystyle\lim_{t\to-\infty} \int_t^0 e^x\,dx = \lim_{t\to-\infty} e^x\Big]_t^0 = \lim_{t\to\infty} e^0 - e^t = \lim_{t\to-\infty} 1 - e^t$

$= 1 - 0 = 1$

evaluate b $\displaystyle\lim_{t\to\infty} \int_0^t e^x\,dx = \lim_{t\to\infty} e^x\Big]_0^t = \lim_{t\to\infty} e^t - e^0 = \lim_{t\to\infty} e^t - 1$

$= \infty - 1 = \infty$

So, $1 + \infty = \infty$; divergent.