5.6 Related Rates

Standards:

MCD2

MCD2b

$\boxed{\text{Old}}$ Implicit Differentiation

① $\frac{d}{dx}\left(x^2y + xy^2 = 3x\right)$

$\left[(x^2)\cdot(1)y' + (y)(2x)\right] + \left[(x)\cdot(2y)y' + (y^2)(1)\right] = 3$

$x^2y' + 2xy + 2xyy' + y^2 = 3$

$x^2y' + 2xyy' + 2xy + y^2 = 3$

$x^2y' + 2xyy' = 3 - 2xy - y^2$

$y'(x^2 + 2xy) = 3 - 2xy - y^2$

$$\boxed{y' = \frac{3 - 2xy - y^2}{x^2 + 2xy}}$$

② $\frac{d}{dx}\left(xy + 2x + 3x^2 = 4\right)$

$\left[(x)(1)\cdot y' + (y)(1)\right] + 2 + 6x = 0$

$xy' + y + 2 + 6x = 0$

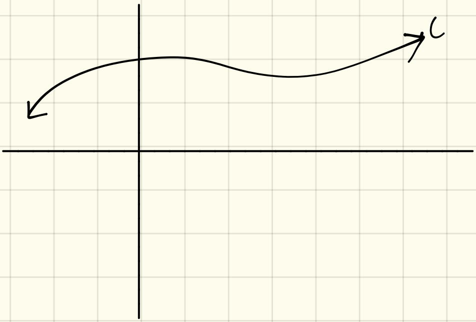
$xy' = -y - 2 - 6x$

$$\boxed{y' = \frac{-y - 2 - 6x}{x}}$$

$\boxed{\text{new}}$ Related Rates

- These are word problems that involve dealing with variables that change over time.
- There is a "twist" to the technique of implicit differentiation in this section.

Let's consider that a particle $P(x,y)$ is moving along a curve C in the plane so that its coordinates x and y are differentiation functions of time t. If D is the from the origin, then find an equation relating $\frac{dD}{dt}$, $\frac{dx}{dt}$ and $\frac{dx}{dt}$.



$$D=\sqrt{x^2+y^2}$$

$$\frac{dD}{dt}=\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x\frac{dx}{dt}+2y\frac{dy}{dt}$$

(Another Example) Finding Related Rates Equations

Assume that radius r of a sphere is a differentiatiable function of t and let V be the volume of the sphere. Find an equation that relates $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

Volume of sphere: $V=\frac{4}{3}\pi r^3$

$$\frac{dV}{dt}=\frac{4}{3}\pi\, 3r^2\frac{dr}{dt}$$

$$=4\pi r^2\frac{dr}{dt}$$

Tips / Strategy for solving Related Rates Problems

1. Understand the problem, identify the variable(s) whose rate of change you seek & variables rate of change you know.
2. Sketch a picture of the situation. Label variables.
3. Write an equation relating the variable whose rate of change you seek with the variable(s) whose rate of change you know. (FORMULA is often GEOMETRIC)
4. Differentiate both sides of the equation implicity with respect to time t.
5. Substitute given quantities from the problem.
6. Interpret the solution.

[Example 1] A is the area of the circle of radius r where the circle is expanding over time. If the radius increases at a constant rate of 1 m/s, how fast is the area increasing when the radius is 30m?

givens:

$\frac{dr}{dt} = 1\,m/s$, $r = 30m$, $\frac{dA}{dt} = ?$

Equation:

$A = \pi r^2$

Differentiate Equation with respect to time t:

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2r \frac{dr}{dt} \pi$$

(evaluate with givens)

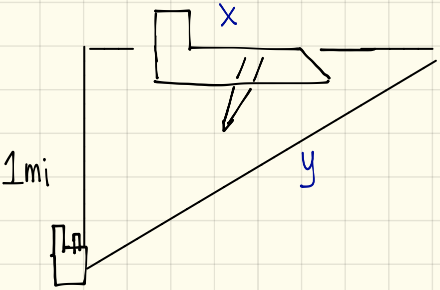
$$? = 2\pi(30m) \cdot (1\,m/s)$$

$$\frac{dA}{dt} = 60\pi\ m^2/s$$

The Area is increasing at $60\pi \frac{m^2}{s}$ when radius is 30m and increasing at 1 m/s.

A Delta airplane is flying horizontally at a speed of 500 $^{mi}/_{h}$ at an altitude of 1 mi. Find the rate at which the distance from the plane to the radar is increasing when it's 2 miles away from the radar station.

Let x = plane horizontal distance from radar at time t.
Let y = direct distance from radar to plane at time t.



givens: $\frac{dx}{dt} = 500\ ^{mi}/_{h}$, $y = 2$ miles, $\frac{dy}{dt} = ?$

Equation:
$$x^2 + 1^2 = y^2$$

Find x:
$$x^2 + (1\text{mi})^2 = (2\text{mi})^2$$
$$x^2 = 1\text{mi}^2 + 4\text{mi}^2$$
$$x^2 = 5\text{mi}^2$$
$$x = \sqrt{5}\ \text{mi}$$

Differentiate Equation with respect to time t:
$$2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$$
$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$
(evaluate with givens)
$$2(\sqrt{5}\text{mi})(500\ ^{mi}/_{h}) = 2(2\text{mi})\frac{dy}{dt}$$
$$1000\sqrt{5}\ \text{mi}^2/h = 4\text{mi}\ \frac{dy}{dt}$$
$$250\sqrt{5}\ \frac{\text{mi}}{h} = \frac{dy}{dt}$$