6.1 Introduction to Sequences

$\boxed{\text{Old}}$ Evaluating Limits

Let's recall the differences between:

$\lim_{x\to a} f(x) = L$ as x-value approaches a, the y-values approaches L.

$\lim_{x\to\infty} f(x) = L$ as x goes toward infinity (∞), the y-values approaches a horizontal asymptote

When evaluating $\lim_{x\to a} f(x) = L$, you can either use:
- expanding
- factor
- common denominator
- multiply by conjugate
- L'Hospital's Rule*

When evaluating $\lim_{x\to\infty} f(x) = L$, you can either use:
- L'Hospital's Rule
- multiply numerator/denominator by degree in the denominator

[Remember $\lim_{x\to\infty} \frac{1}{x} = 0$.

[Examples] Evaluate.

① $\lim_{x\to 1} \frac{x^2+x-2}{x-1} = \frac{0}{0}$

$$\lim_{x\to 1} \frac{x^2+x-2}{x-2} \underset{=}{L'H} \lim_{x\to 1} \frac{2x+1}{1} = \frac{2(1)+1}{1} = 3.$$

② $\lim_{x\to\infty} \frac{x^2+x-2}{x-1}$

$$\lim_{x\to\infty} \frac{x^2+x-2}{x-1} \underset{=}{L'H} \lim_{x\to\infty} \frac{2x+1}{1} = \lim_{x\to\infty} 2x+1 = \infty, \text{ divergent.}$$

③ $\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2}$

$$\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2} \overset{L'H}{=} \lim_{x\to\infty} \frac{4x+1}{2x+1} \overset{L'H}{=} \lim_{x\to\infty} \frac{4}{2} = 2.$$

$\boxed{\text{new}}$ Introduction to Sequences

A sequence is an infinite, ordered list of numbers:

$a_1, a_2, a_3, a_4, \ldots$ (sometimes it come start with index 0)

[Notation of Sequences]

- a_n - where $\underline{n}$ is referring to the term number in the sequence.

(Example) $12, 24, 36, 48, \ldots$

$\ \ \downarrow\ \ \ \downarrow\ \ \ \downarrow\ \ \ \downarrow$

$\ a_1\ \ a_2\ \ a_3\ \ a_4$

- Brace Notation is often used to represent sequences:

$\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$.

(Example) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ represented as $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

(Example) $\left\{\frac{(-1)^{n-1}}{n^2}\right\}_{n=1}^{\infty}$ The terms of the sequence are: $1, \frac{-1}{4}, \frac{1}{9}, \frac{-1}{16}, \ldots$

FACT The values in the range are called the term of the sequence.

Domain: 1 2 3 4 5 . . . n
(position in the sequence)

Range: a_1 a_2 a_3 a_4 a_5 . . . a_n
(the actual sequence)

(Example) For the sequence 12, 24, 36, 48, ..., Label the domain & range.

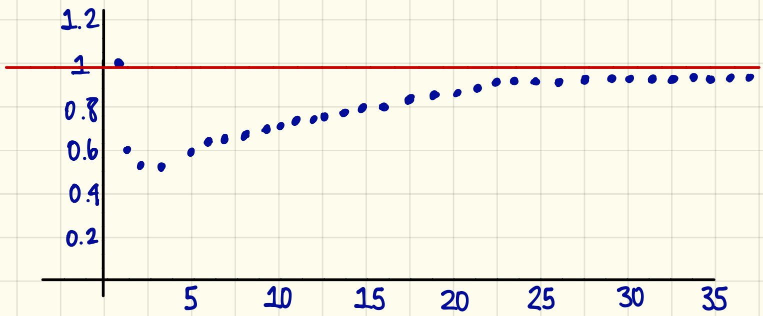
Domain	1	2	3	4	...
Range	12	24	36	48	...

which can be written as ordered pair:
(1,12), (2,24), (3,36), (4,48), ...

[The Limit of a Sequence]

Let's consider the sequence $\left\{\frac{n^2+5}{n^2+5n}\right\}_{n=1}^{\infty}$. Determine $\lim_{n\to\infty} a_n$.

$$\left\{\frac{n^2+5}{n^2+5n}\right\}_{n=1}^{\infty} = 1, \frac{9}{14}, \frac{7}{12}, \frac{7}{12}, \frac{3}{5}, \frac{41}{66}, \frac{9}{14}, \frac{69}{104}, \frac{43}{63}, \frac{7}{10}, \frac{63}{88}, \ldots$$



$$\lim_{n\to\infty}\frac{n^2+5}{n^2+5n} = \lim_{x\to\infty}\frac{x^2+5}{x^2+5x} \overset{L'H}{=} \lim_{x\to\infty}\frac{2x}{2x+5} \overset{L'H}{=} \lim_{x\to\infty}\frac{2}{2} = 1$$

Definition

Suppose that $a_n = f(n)$, where f is the function defined for $x \geq 1$.

If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Special Notes about Sequences:

- A sequence converges if $\lim_{n\to\infty} a_n = L$ exist. Otherwise, it diverges.

$\boxed{\text{OR}}$

$\lim_{n\to\infty} (-1)^n = \text{d.n.e}$ — It's bouncing back & forth from 1 to -1; and it's not approaching anything.

- A sequence is bounded if $a_n \leq M$ (above) or if $M \leq a_n$ (below) — never going to be above or below a certain value.

- A monotonic sequence is always increasing ($a_1 < a_2 < a_3 < a_4 \ldots$) or always decreasing ($a_1 > a_2 > a_3 > a_4 \ldots$)

[Examples] Determine whether the sequence converges or divergences.

① $\left\{\frac{n+1}{3n-1}\right\}_{n=1}^{\infty}$ $\lim_{n\to\infty} \frac{n+1}{3n-1} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$; convergent.

② $\left\{\frac{n}{1+\sqrt{n}}\right\}_{n=1}^{\infty}$ $\lim_{n\to\infty} \frac{n}{1+\sqrt{n}} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{2}n^{-\frac{1}{2}}} = \lim_{n\to\infty} 2\sqrt{n} = \infty$; divergent.

$\boxed{\text{note}}$ You may have to use the Squeeze Theorem to evaluate.

Let's recall the Squeeze Theorem:

$\boxed{\text{Squeeze Theorem}}$

If $g(x) \le f(x) \le h(x)$ for all $x \ne a$ in some interval about a, and

$$\lim_{x\to a} g(x) = \lim_{x \to a} h(x) = L,$$

then $\displaystyle\lim_{x\to a} f(x) = L$.

$\boxed{\text{Examples}}$ Determine ~~ne~~ whether the sequence is convergent or divergent.

1. $\left\{\dfrac{n-3(-1)^n}{n+2}\right\}_{n=1}^{\infty}$

when n is odd:

$$\lim_{n\to\infty}\frac{n-3}{n+2} \overset{\text{L'H}}{=} \lim_{n\to\infty}\frac{1}{1} = 1$$

when n is even

$$\lim_{n\to\infty}\frac{n+3}{n+2} \overset{\text{L'H}}{=} \lim_{n\to\infty}\frac{1}{1} = 1$$

Since, $\dfrac{n-3}{n+2} \le \dfrac{n-3(-1)^n}{n+2} \le \dfrac{n+3}{n+2}$ and $1 \le \dfrac{n-3(-1)^n}{n+2} \le 1$, then

$\displaystyle\lim_{n\to\infty}\frac{n-3(-1)^n}{n+2} = 1$ because of the Squeeze Theorem.

② $\left\{ \frac{(-1)^n n^3}{n^3+2n^2+1} \right\}_{n=1}^{\infty}$

when n is odd:

$$\lim_{n\to\infty} \frac{-n^3}{n^3+2n^2+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{-3n^2}{3n^2+4n} \overset{L'H}{=} \lim_{n\to\infty} \frac{-6n}{6n+4} \overset{L'H}{=} \lim_{n\to\infty} \frac{-6}{6} = -1$$

when n is even:

$$\lim_{n\to\infty} \frac{n^3}{n^3+2n^2+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{3n^2}{3n^2+4n} \overset{L'H}{=} \lim_{n\to\infty} \frac{6n}{6n+4} \overset{L'H}{=} \lim_{n\to\infty} \frac{6}{6} = 1$$

$$\lim_{n\to\infty} \frac{(-1)n^3}{n^3+2n^2+1} = \infty;$$ divergent.

[Example] Consider $\left\{ \frac{2n-3}{3n+4} \right\}$. Prove that a_n either increases or decreases.

Sidework:

$\left(\frac{1}{7}, \ldots, \frac{2}{3}\right) \to$ converges at $\frac{2}{3}$ show that $a_n < a_{n+1}$ (sequence increases)

$$a_n < a_{n+1}$$

$$\frac{2n-3}{3n+4} < \frac{2(n+1)-3}{3(n+1)+4}$$

$$\frac{2n+3}{3n+4} < \frac{2n-1}{3n+7}$$

$$\iff (2n+3)(3n+7) < (2n-1)(3n+4)$$

$$\iff \cancel{6n^2} \cancel{+5n} - 21 < \cancel{6n^2} \cancel{+5n} - 4$$

$-21 < -4$; the sequence is increasing.

Homework page 441] 31-39