6.2 Area Between Curves

Riemann Sums

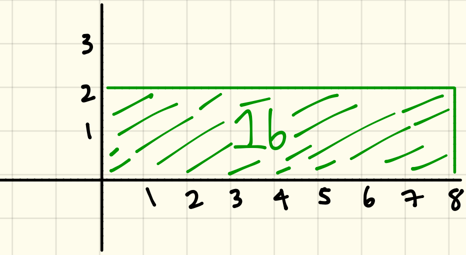
Standards:

MC11

MC11b

$\boxed{\text{Old}}$ Area of Rectangles

Let's consider the following situation:
Suppose you travelled at $2\,\frac{ft}{sec}$ for 8 seconds. How far did you go?



Answer: 16 feet.

Why?

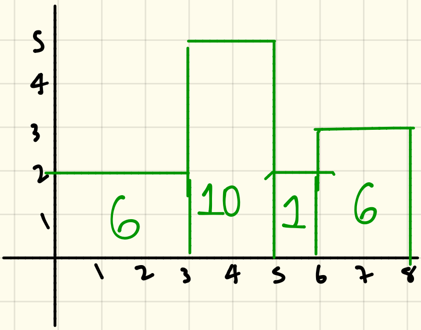
The rate is $2\frac{ft}{sec}$ and 8 seconds of travel time.

$$\text{Distance} = \text{Rate} \cdot \text{Time}$$
$$= 2\tfrac{ft}{sec} \quad 8 \text{ secs}$$
$$= 16 \text{ feet.}$$

Conclusion

The area of rectangles gives the <u>total distance travelled.</u>

Let's consider a <u>slightly</u> realistic scenario:



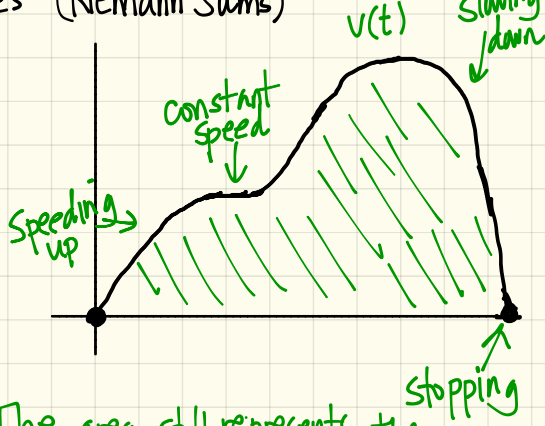
So the total distance travelled:

$$6 + 10 + 1 + 6 = 23 \text{ feet.}$$

$\boxed{\text{new}}$ Area & Distances under curves (Riemann Sums)

What would be a <u>more</u> realistic situation?

A velocity function that varies continuously with time.



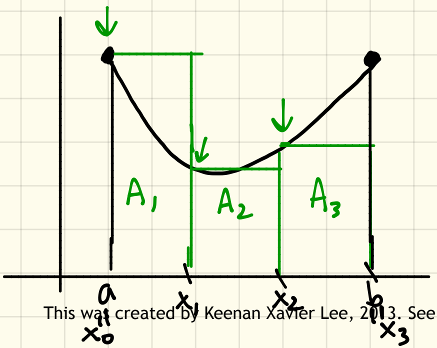
<u>Dilemma</u>: But how do we find the area under this curve?

The area still represents the total distance travelled.

Basic Idea: We need to understand how to <u>compute</u> area of curved regions. We do this by "approximating" the region with <u>rectangles</u> and adding their areas (Riemann sums).

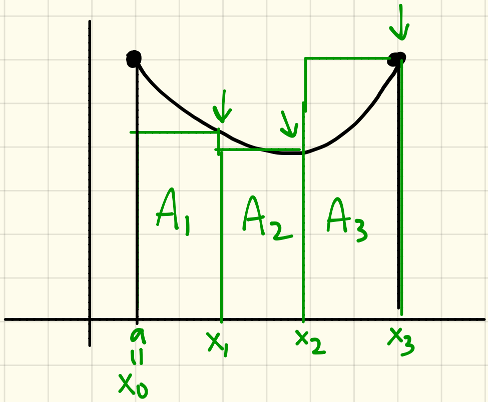
You can use <u>Right Endpoint Approximation</u>, <u>Left Endpoint Approximation</u>, and <u>Midpoint Approximation</u>.

<u>Left Rectangular Approximation Method</u> (LRAM)



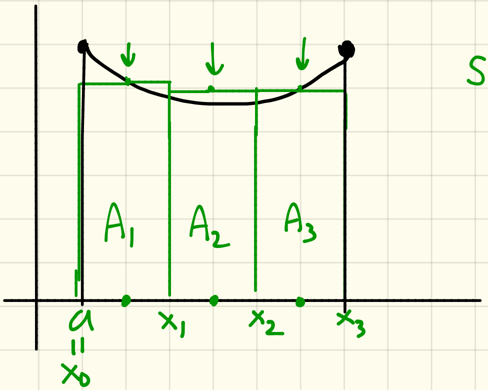
$$\text{Sum} = A_1 + A_2 + A_3$$
$$= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2)$$
$$= \Delta x\left[f(x_0) + f(x_1) + f(x_2)\right]$$

Right Rectangular Approximation Method (RRAM)



$$\text{Sum} = A_1 + A_2 + A_3$$
$$= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)(x_3 - x_2)$$
$$= \Delta x\left[f(x_1) + f(x_2) + f(x_3)\right]$$

Midpoint Rectangular Approximation Method (MRAM)



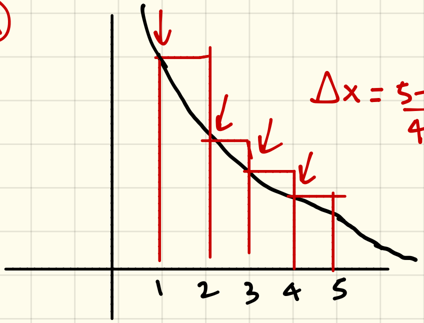
$$\text{Sum} = A_1 + A_2 + A_3$$
$$= f\left(\frac{x_0+x_1}{2}\right)(x_0 - x_1) + f\left(\frac{x_2+x_1}{2}\right)(x_2 - x_1) + f\left(\frac{x_3+x_2}{2}\right)(x_3 - x_2)$$
$$= \Delta x\left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_2+x_1}{2}\right) + f\left(\frac{x_3+x_2}{2}\right)\right].$$

$$\Delta x = \frac{b-a}{n}$$

a = start
b = stop
n = # of rectangles

[Example] Find the area using $f(x)=\frac{1}{x}$ from $x=1$ to $x=5$ using 4 rectangles using (a) RRAM, (b) LRAM and (c) MRAM.

(a)



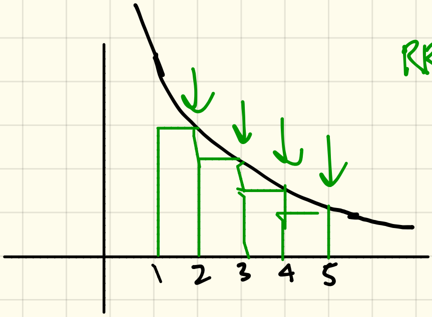
$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1$

$$\text{LRAM sum} = \Delta x\left[f(1)+f(2)+f(3)+f(4)\right]$$

$$= 1\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$$

$$\approx 2.083$$

(b)

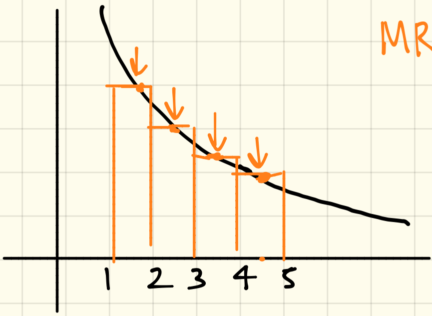


$$\text{RRAM sum} = \Delta x\left[f(x_2)+f(x_3)+f(x_4)+f(x_4)\right]$$

$$= 1\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)$$

$$\approx 1.283$$

(c)



$$\text{MRAM sum} = \Delta x\left[f(1.5)+f(2.5)+f(3.5)+f(4.5)\right]$$

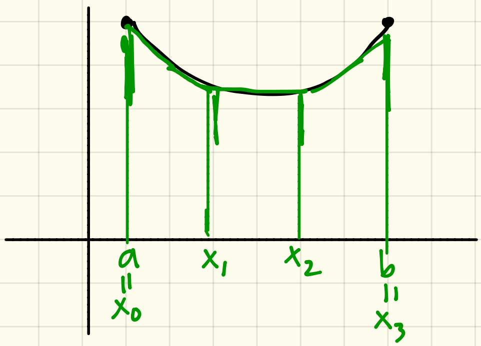
$$= 1\left[\frac{1}{1.5}+\frac{1}{2.5}+\frac{1}{3.5}+\frac{1}{4.5}\right]$$

$$\approx 1.575$$

LRAM is an overapproximation & RRAM is an underapproximation.

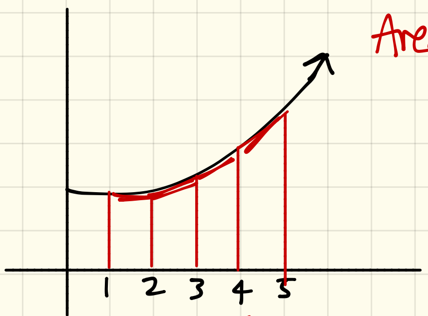
$\boxed{\text{Note:}}$ The more rectangles applied, the more areas being computed, and the accurate the approximation of the area under the curve will be.

Trapezoidal Rule for Approximation



$$\text{Area} = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

[Example] Approximate using Trapezoidal Rule for $f(x) = 1 + x^2$ using $n = 4$ (4 rectangles) from $x = 1$ to $x = 5$



$$\text{Area} = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2(x_3) + 2(x_4) + f(x_5)\right]$$

$$= \frac{1}{2}\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right]$$

$$= \frac{1}{2}\left[2 + 2(5) + 2(10) + 2(17) + 26\right]$$

$$= 46$$

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = \frac{4}{4}$$