6.2 Introduction to Series

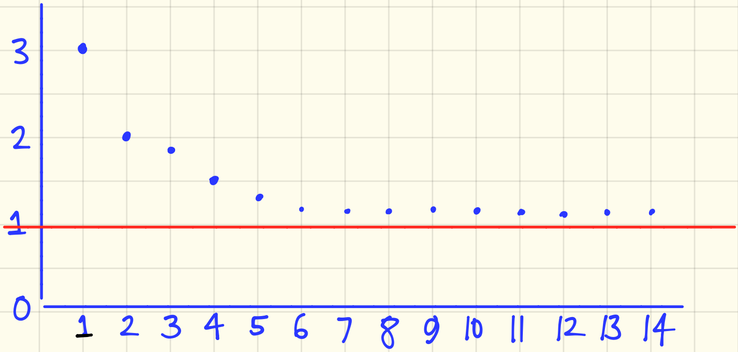
$\boxed{\text{old}}$ Sequences

Consider $\left\{\frac{n+2}{n}\right\}_{n=1}^{\infty}$. Provide a graphical, analytical & algebraic representation. Also, determine if it converges or divergences.

[ANALYTICAL]

Domain	1	2	3	4	5	6	•	•	•
Range	3	2	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{4}{3}$	•	•	•

[GEOMETRIC]



[ALGEBRAIC]

$$\lim_{n\to\infty} \frac{n+1}{n} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1}{1} = 1.$$

[VERBAL]

$\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$ converges at $y=1$.

$\boxed{\text{New}}$ Series

Let's consider $\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$. Express the sequence.

Sequence of terms: $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \frac{7}{128}, \frac{8}{256}, \ldots, \frac{2}{2^{20}}, \ldots$

<u>Definition</u>: An <u>infinite series</u> is adding of all terms for a given sequence.

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

(Example) Using consider example, express the infinite series.

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \frac{7}{128} + \frac{8}{256} + \cdots + \frac{2}{2^{20}} + \cdots = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

But, does it make sense to discuss about the sum of infinitely many terms?

Dilemma: How do we express finite sums of series & determine limits of series?

Let's find some finite sums:

n	1	2	3	4	5	6	7	8	. . . 20 . . .
$\sum$	$\frac{1}{2}$	1	$\frac{11}{8}$	$\frac{13}{8}$	$\frac{57}{32}$	$\frac{15}{8}$	$\frac{247}{128}$	$\frac{251}{128}$	. . . 1.99998 . . .

<u>Definition</u>: A <u>partial sum</u> is adding finite sums.

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots + a_n = \sum_{n=1}^{n} a_i$$

Looking at the table, as n approaches ∞, the sum seems to approach 2.

conclusion If the sequence of partial sums converge to a value (L), then the series will converge to that value.

FACT $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{k} a_k$.

Basically, if $\lim_{n\to\infty} S_n = L$, then $\sum_{n=1}^{\infty} a_n = L$.

[Examples] Determine where the summation converges.

① $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{n(n+1)} + \cdots$$

sequence of partial sum: $S_1, S_2, S_3, S_4, S_5, \ldots$

$$S_n = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1} = 1.$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

② $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1}$

$$\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right) + \cdots$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \underbrace{\frac{1}{n+1}}_{\to 0} = 1.$$

$\boxed{\text{note}}$ $\sum \frac{1}{k} - \frac{1}{k+1}$ is a telescoping series — where the partial sums eventually have a fixed number of terms after the elimination.

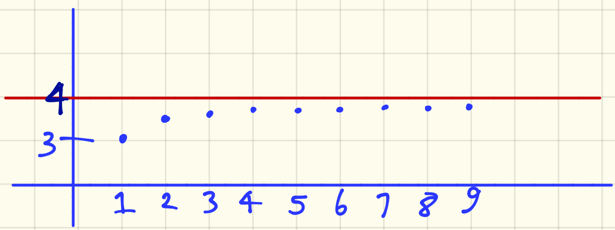
[Geometric Series]

Let's consider a sequence where $a_0 = 3$ and the common ratio $(r) = \frac{1}{4}$.

Sequence: $3, \frac{3}{4}, \frac{3}{16}, \frac{3}{64}, \frac{3}{256}, \frac{3}{1024}, \ldots$

Partial Sum: $3, \frac{15}{4}, \frac{63}{16}, \frac{255}{64}, \frac{1023}{256}, \ldots$

(in decimal) $3, 3.75, 3.9375, 3.984375, 3.99609375, \ldots$



Conclusion: The (geometric) series seems as though it approaches 4

But there is no proof!

Dilemma: How do we develop a "closed" formula or the equation for partial sum for geometric series (so we can find the limit)?

Definition: Geometric Series

Given $r \in \mathbb{R}$ and $a_0 \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} a_1 (r)^{n-1} = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots$$

is called a geometric series.

The sum of the series (as always) is the limit of partial sums:

$$\sum_{n=1}^{\infty} a_1 (r)^{n-1} = \lim_{n \to \infty} \left(a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}\right)$$

We can get the "closed" formula for the n^{th} partial sum as follows:

proof:

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}$$

$$- \quad rS_n = -(a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^n)$$ (multiply r & subtract)

$$S_n - rS_n = a_1 - a_1 r^n$$

$$S_n(1-r) = a_1(1-r^n)$$

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad (\text{if } r \neq 1) \quad \blacksquare$$

So, these are the partial sums of a geometric series (if $r \neq 1$)

$$S_n = \sum_{n=1}^{n} a_1 (r)^{n-1} = \frac{a_1(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} = \lim_{n\to\infty} \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1 - \lim_{n\to\infty} a r^n}{1-r}$$

if $r > 1$, then diverges

But, if $-1 < r < 1$, then $\frac{a_1}{1-r}$.

$$\sum_{n=0}^{\infty} a_1 (r)^{n-1} = \frac{a_1}{1-r} \quad \text{if } |r| < 1.$$

Let recall the previous considered example to prove that it converges to 4.

$$\sum_{n=1}^{\infty} a_0 (r)^{n-1} = \sum_{n=1}^{\infty} (3)\left(\tfrac{1}{4}\right)^{n-1} = \frac{3}{1-\frac{1}{4}} = \frac{3}{\frac{3}{4}} = 4$$

[Examples] Determine where the Σ converges.

(1) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{3^k}$

$$\sum_{k=0}^{\infty} \frac{2^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1-\left(\frac{2}{3}\right)} = \frac{1}{\frac{1}{3}} = 3.$$

② $\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^n$

$$\sum_{k=0}^{\infty} 4\left(\frac{1}{4}\right)^n = \frac{4}{1-\left(\frac{1}{4}\right)} = \frac{4}{\frac{3}{4}} = 4 \cdot \frac{4}{3} = \frac{16}{3}.$$

③ $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} = \sum_{k=0}^{\infty} \left(\frac{-1}{2}\right)^k = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

④ $\sum_{k=0}^{\infty} \frac{3^{k+1}-2^k}{5^k}$

$$\sum_{k=0}^{\infty} \frac{3^{k+1}-2^k}{5^k} = \sum_{k=0}^{\infty} \frac{3^{k+1}}{5^k} - \sum_{k=0}^{\infty} \frac{2^k}{5^k}$$

$$= \sum_{k=0}^{\infty} \frac{3^1 3^k}{5^k} - \sum_{k=0}^{\infty} \frac{2^k}{5^k}$$

$$= \sum_{k=0}^{\infty} 3\left(\frac{3}{5}\right)^k - \sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k$$

$$= \frac{3}{1-\left(\frac{3}{5}\right)} - \frac{1}{1-\left(\frac{2}{5}\right)}$$

$$= \frac{3}{\frac{2}{5}} - \frac{1}{\frac{3}{5}} = \frac{35}{6}$$

⑤ $\displaystyle\sum_{k=3}^{\infty} \frac{1}{3^k}$

note: index needs to start at $k=0$.

$$\sum_{k=3}^{\infty} \frac{1}{3^k} = \sum_{k=0}^{\infty} \frac{1}{3^{3+k}} = \sum_{k=0}^{\infty} \frac{1}{3^3 \cdot 3^k} = \sum_{k=0}^{\infty} \frac{1}{27}\left(\frac{1}{3}\right)^k = \frac{\frac{1}{27}}{1-\left(\frac{1}{3}\right)}$$

$$= \frac{1}{18}.$$

[Basic Test for Divergence] The N^{th} Term Test

If the sequence of terms $\{a_k\}_{k=1}^{\infty}$ does not converge to 0, then the series $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges.

[Examples] Convergent or Divergent.

① $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^2-1}$

Test for Divergence: $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2-1} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{2n}{2n} = \lim_{n\to\infty} 1 = 1 (\neq 0)$

Since $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2-1} \neq 0$ then $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^2-1}$ is divergent.

② $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$

$\lim_{n\to\infty} \frac{n^2}{5n^2+4} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{10n+4} \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{10} = \frac{1}{5} (\neq 0)$

$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is divergent.

WARNING If $\lim_{n\to\infty} a_n = 0$, then we can't conclude whether convergent or divergent

③ $\sum_{K=1}^{\infty} \frac{1}{K}$

$\lim_{n\to\infty} \frac{1}{K} = 0 \longrightarrow$ can't conclude on convergence or divergence.

$\sum_{K=1}^{\infty} \frac{1}{K} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ (The Harmonic Series)

$S_1 = 1$ $\quad S_2 = 1 + \frac{1}{2}$ $\quad S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$

$\left(= 1 + \frac{2}{2}\right)$

$S_8 = 1 + \frac{1}{2}\left(+ \frac{1}{3} + \frac{1}{4}\right)\left(+ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2}\left(+ \frac{1}{4} + \frac{1}{4}\right)\left(+ \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$

$\left(= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right)$

$\left(= 1 + \frac{3}{2}\right)$

$$S_{16} = 1 + \frac{1}{2} \left(+ \frac{1}{3} + \frac{1}{4} \right) \left(+ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) \left(+ \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \right)$$

$$> 1 + \frac{1}{2} \left(+ \frac{1}{4} + \frac{1}{4} \right) \left(+ \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} \right)$$

$$\left(= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right)$$

$$\left(= 1 + \frac{4}{2} \right)$$

$S_{32} > 1 + \frac{5}{2}$ $\qquad$ $S_{64} > 1 + \frac{6}{2}$

So, ... $S_{2^n} > 1 + \frac{n}{2}$

$$\lim_{n \to \infty} 1 + \frac{n}{2} = \lim_{n \to \infty} 1 + \frac{1}{2}n = \infty; \text{ divergent.}$$

SUMMARY for Convergence or Divergence

Convergence	Divergence
1. Geometric Series: if $\lvert r \rvert < 1$ for $a_n = a(r)^n$, then $\sum a_n$ converges at $\frac{a}{1-r}$.	1. Geometric Series if $\lvert r \rvert > 1$ for $a_n = a(r)^n$, then $\sum a_n$ diverges.
2. N^{th} Term Test:	2. N^{th} Term Test: if $\lim_{n \to \infty} a_n \neq 0$, then $\sum a_n$ diverges.

Homework page 481] 7-10, 11-19 (Radius of Convergence)

page 511] 29-30, 32, 34, 38-41 (Geometric Series & N^{th} Term Tests)