6.3 The Fundamental Theorem of Calculus

Standards:

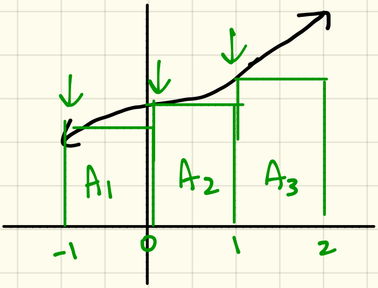
MC11

MC11b

$\boxed{\text{Old}}$ Areas & Distances

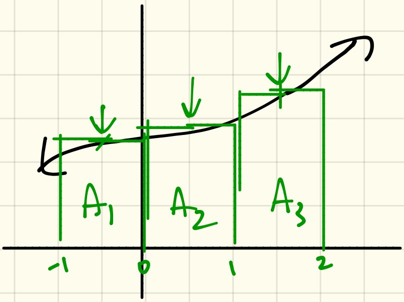
Let's consider the function $f(x) = x^3 + 2$. Approximate the area from $x = -1$ to $x = 2$ using 3 rectangles.

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{3} = \frac{3}{3} = 1$$



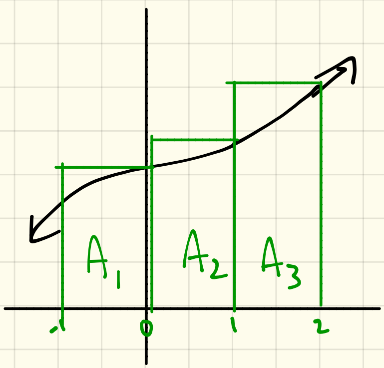
LRAM

$$\text{Sum} = A_1 + A_2 + A_3$$
$$= \Delta x \left[f(-1) + f(0) + f(1) \right]$$
$$= 1\left[1 + 2 + 3\right]$$
$$= 6$$



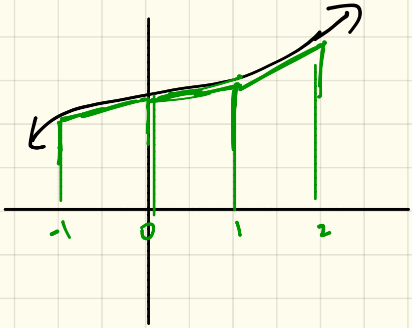
MRAM

$$\text{Sum} = A_1 + A_2 + A_3$$
$$= \Delta x \left[f(-.5) + f(.5) + f(1.5) \right]$$
$$= 1\left[1.875 + 2.125 + 5.375\right]$$
$$= 9.375$$



RRAM

$$\text{Sum} = A_1 + A_2 + A_3$$
$$= \Delta x \left[f(0) + f(1) + f(2) \right]$$
$$= 1\left(2 + 3 + 10\right)$$
$$= 15$$



Trapezoidal Rule Approx

$$\text{Sum} = A_1 + A_2 + A_3$$
$$= \Delta x\left[f(-1) + 2f(0) + 2f(1) + f(2)\right]$$
$$= 1\left[1 + 2 + 3 + 10\right]$$
$$= 16$$

$\boxed{\text{Also old}}$ Integration → Antidifferentiation

① $\int x^5 + 3x^3 + \frac{2}{x^2} + \pi \, dx = \int x^5 + 3x^3 + 2x^{-2} + \pi \, dx$

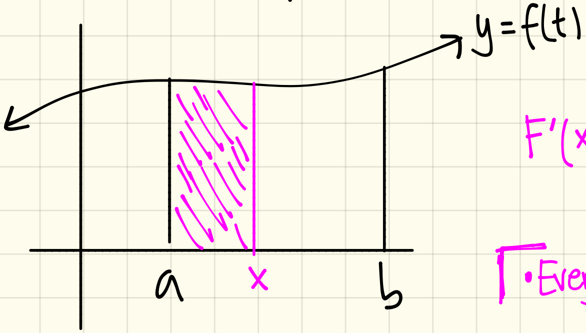
$$= \frac{x^6}{6} + \frac{3x^4}{4} + \frac{2x^{-1}}{-1} + \pi x + C$$
$$= \frac{1}{6}x^6 + \frac{3}{4}x^4 - \frac{2}{x} + \pi x + C$$

② $\int \frac{t^3 + 2t^2}{\sqrt{t^2}} dt = \int \frac{t^3}{\sqrt{t}} + \frac{2t^2}{\sqrt{t}} dt = \int \frac{t^3}{t^{1/2}} + \frac{2t^2}{t^{1/2}} dt$

$$= \int t^{5/2} + 2t^{3/2} dt = \frac{t^{7/2}}{7/2} + \frac{2t^{5/2}}{5/2} + C$$
$$= \frac{2}{7}t^{7/2} + \frac{4}{5}t^{5/2} + C$$

$\boxed{\text{New}}$ The FTC

Let's consider a function f continuous on $[a,b]$.



1^{st} part of the FTC:

$$F'(x) = \frac{d}{dt}\int_a^x f(t)\,dt = f(x)$$

- Every continuous f has an antiderivative $F(x)$
- connection between differentiation & integration.

$F(x) = \int_a^x f(t)\,dt$, where x is in $[a,b]$

Definition

If f is continuous on $[a,b]$ then, $F(x) = \int_a^x f(t)\,dt$, $a \le x \le b$.

is continuous on $[a,b]$, differentiable on (a,b) & $F'(x) = f(x)$.

[Example] Find derivatives.

① $g(x) = \int_1^x (t^2-1)^{20}\,dt$

$$g'(x) = \frac{d}{dx}\int_1^x (t^2-1)^{20}\,dt = (x^2-1)^{20}$$

② $f(x) = \int_{\pi}^{x} \frac{\cos^2 t}{\ln(t-\sqrt{t})}\,dt$

$$f'(x) = \frac{d}{dt}\int_{\pi}^{x} \frac{\cos^2 t}{\ln(t-\sqrt{t})}\,dt = \frac{\cos^2 x}{\ln(x-\sqrt{x})}$$

Sometimes you might have to use the chain rule...

③ $f(x) = \int_{\pi}^{x} \cot^2 t\,dt$

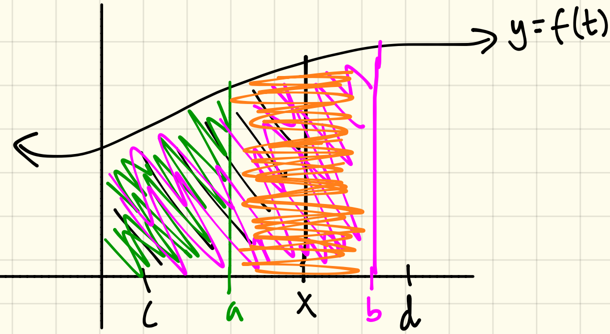
$$f'(x) = \frac{d}{dx}\int_{\pi}^{x} \cot^2 t\,dt = \cot^2 x$$

④ $h(x) = \int_{\pi}^{x^2} \cot^2 t\,dt$

$$h'(x) = \frac{d}{dx}\int_{\pi}^{x^2} \cot^2 t\,dt = \cot^2(x^2)\cdot 2x$$

$$= 2x\cot^2(x^2).$$

Let's consider a function f continuous on $[c, d]$.



1st part of FTC

$$F(x) = \int_c^x f(t)\,dt$$

$$F'(x) = f(x)$$

Let's think about what: (assuming $b > a$)

$$F(b) - F(a) = \int_c^b f(t)\,dt - \int_c^a f(t)\,dt = \int_a^b f(t)\,dt$$

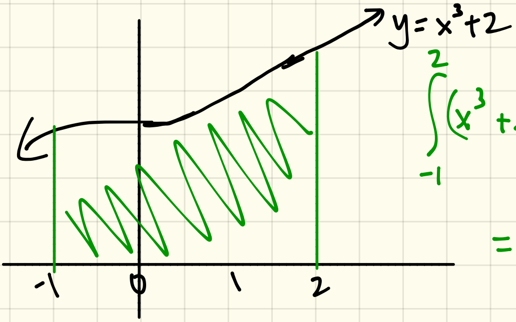
2nd part of FTC

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a) \text{ where } F' = f.$$

(so where F is the antiderivative of f)

[Example] – back to old example of Area & Distances

NOW WE ARE GOING TO COMPUTE the area.



$$\int_{-1}^{2} (x^3+2)\,dx = \frac{x^4}{4} + 2x \Big]_{-1}^{2}$$

$$= \left[\frac{(2)^4}{4} + 2(2)\right] - \left[\frac{(-1)^4}{4} + 2(-1)\right]$$

$$= [8] - [-1.75]$$

$$= 9.75$$

② $$\int_{0}^{2} 2 - x^2\,dx = 2x - \frac{x^3}{3}\Big]_{0}^{3} = \left[2(2) - \frac{(2)^3}{3}\right] - \left[2(0) - \frac{(0)^3}{3}\right]$$

$$= \left[4 - \frac{8}{3}\right] - [0]$$

$$\approx 1.333$$

③ $$\int_{2}^{5} 2 + 3x - x^2\,dx = 2x + \frac{3x^2}{2} - \frac{x^3}{3}\Big]_{2}^{5} =$$

$$= \left[2(5) + \frac{3(5)^2}{2} - \frac{(5)^3}{3}\right] - \left[2(1) + \frac{3(1)^2}{2} - \frac{(1)^3}{3}\right]$$

$$\approx 2.666$$