6.3 The Integral Test

$\boxed{\text{old}}$ Improper Integrals

Let's recall the idea behind Infinite Integrals:
The idea is to find the area when one side is unbounded (or both sides are unbounded).

Infinite Integrals, where an infinity is either at one or both limits of integration

↳ (one case) If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$.

[Examples] Integrate.

① $\int_1^\infty \frac{dx}{x^3} = \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^t x^{-3}\,dx = \lim_{t\to\infty} \frac{x^{-2}}{-2}\Big]_1^t$

$= \lim_{t\to\infty} \frac{-1}{2x^2}\Big]_1^t = \lim_{t\to\infty} \left[\frac{-1}{2t^2} - \frac{-1}{2(1)}\right] = \lim_{t\to\infty} \left[\underbrace{\left(\frac{-1}{2t^2}\right)}_{\to 0} + \frac{1}{2}\right]$

$= 0 + \frac{1}{2} = \frac{1}{2}$, convergent.

② $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t = \lim_{t\to\infty} \left[\ln|t| - \ln|1|\right]$

$= \lim_{t\to\infty} \left[\underbrace{\ln|t|}_{\to\infty} - 0\right] = \infty - 0 = \infty$; divergent.

$\boxed{\text{new}}$ The Integral Test

Let's consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Determine where the series convergences or divergences.

Recall from 6.2 — if If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n\to\infty} a_n$ divergences.

But warning... if $\lim_{n\to\infty} a_n = 0$, then we can't conclude on divergence or convergence.

Test for Divergence:

$$\lim_{n\to\infty} \frac{1}{n^2} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{0}{2n} = 0.$$

Since $\lim_{n\to\infty} \frac{1}{n^2} = 0$, then we can't conclude if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is divergent or convergent.

Dilemma: After this technique, what other alternative can I use to test for divergence or convergence?

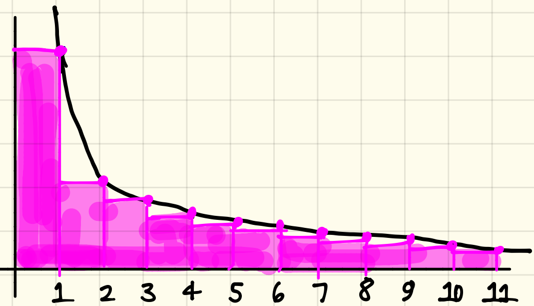
[The Integral Test]

If $a_k = f(k)$, where f is :
1) positive (above x-axis)
2) continuous (no breaks in graphs in interval)
3) decreasing ($f' < 0$)

on $[1, \infty)$ then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx \text{ either both converge or diverge.}$$

Convergence Illustration

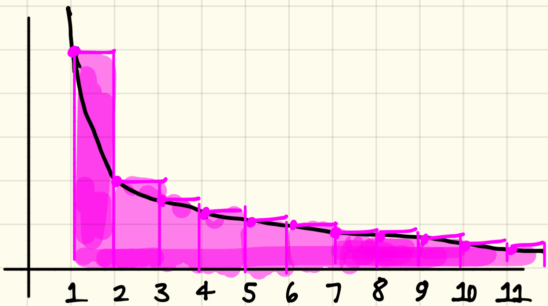


$$\sum_{k=2}^{\infty} a_k < \int_1^{\infty} f(x)\,dx$$

(if the improper integral is finite, then the series is finite)

$\Longrightarrow$ convergent

Divergence Illustration



$$\sum_{k=2}^{\infty} a_k > \int_1^{\infty} f(x)\,dx$$

(infinite region)

$\Longrightarrow$ divergent

[p-series]

Suppose we have the series of the form:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots, \quad \text{where } p > 0.$$

case 1 when $p=1$: (The Harmonic Series)

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Let's evaluate the improper integral to determine whether converge or diverge:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x \Big]_1^t = \lim_{t\to\infty} [\ln t - \ln 1]$$

$$= \lim_{t\to 0} \ln t = \infty; \text{ divergent.}$$

So, $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

Case 2 when $p \neq 1$:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots, \quad \text{where } p > 0.$$

Let's evaluate the improper integral to determine whether converge or diverge:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left. \frac{x^{p+1}}{p+1} \right]_1^t$$

$$= \lim_{t\to\infty} \frac{1}{p+1} t^{p+1} - 1$$

conclusion

- If $p > 1$ — $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges

- If $p \leq 1$ — $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges

$\boxed{\text{Warning}}$ We don't know what value the series converges to. (if it converges)

[Example] Determine whether the series is convergent or divergent.

① $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k} = \int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x} = \lim_{t\to\infty} \int_2^{t} \frac{1}{u}\,du = \lim_{t\to\infty} \ln|u|\Big]_2^t$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$= \lim_{t\to\infty} \ln|\ln x|\Big]_2^t = \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2| = \infty;\ \text{divergent.}$$

$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ is divergent.

② $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} = \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{u^2}\,du = \lim_{t\to\infty} \frac{-1}{u}\Big]_2^t$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$= \lim_{t\to\infty} -\frac{1}{\ln x}\Big]_2^t = \lim_{t\to\infty}\left[\underbrace{\frac{-1}{\ln x}}_{\to 0} - \frac{-1}{\ln 2}\right] = \frac{1}{\ln 2};\ \text{convergent}$$

$\sum_{k=1}^{\infty} \frac{1}{k \ln k}$ is convergent.

③ $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_1^{t} \frac{1}{u}\,du$$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= \lim_{t\to\infty} \ln|u| \Big]_1^t = \lim_{t\to\infty} \ln|2x+1| \Big]_1^t = \lim_{t\to\infty} \underbrace{\ln|2t+1|}_{\to\infty} - \ln|3| = \infty \therefore \text{ divergent}$$

$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent.

SUMMARY for Convergence or Divergence

Convergence

1. Geometric Series:

if $|r|<1$ for $a_n = a(r)^n$, then $\sum a_n$ converges at $\frac{a}{1-r}$.

2. N^{th} Term Test

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3. p-series:

if $p>1$, $\sum \frac{1}{k^p}$ converges

4. The Integral Test:

if $\int_1^\infty f(x)dx$ converges, then $\sum a_k$ converges

Divergence

1. Geometric Series:

if $|r|>1$ for $a_n = a(r)^n$, then $\sum a_n$ diverges.

2. N^{th} Term Test:

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

3. p-series:

if $p \leq 1$, $\sum \frac{1}{k^p}$ diverges

4. The Integral Test:

if $\int_1^\infty f(x)dx$ diverges, then $\sum a_k$ diverges.

Homework page 523] 1-3, (Quick Review) 1-2, 8-10, 14 (Section Exercise Review)