# 6.4 The Comparison Test

# Old Convergence versus Divergence

[Examples] Determine whether the series converges or diverges.

① $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ (Geometric Series)

$r = \frac{2}{3} < 1$; converges.

② $\sum_{n=2}^{\infty} \frac{n^2}{n^2+1}$ ($N^{th}$ Term Test)

$\lim_{n\to\infty} \frac{n^2}{n^2+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{2n} = \lim_{n\to\infty} 1 \neq 0$; diverges

③ $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ ( Geometric Series)

$r = \frac{1}{e} < 1$; converges.

④ $\sum_{n=1}^{\infty} \frac{1}{e^n}$ (or Integral Test)

$\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} e^{-n} = \int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big]_1^t$

$\lim_{t\to\infty} \left[-e^{-t} - -e^{-1}\right] = \lim_{t\to\infty} \underbrace{\frac{-1}{e^t}}_{\to 0} + \frac{1}{e} = \frac{1}{e}$; converges.

⑤ $\sum_{n=1}^{\infty} \frac{1}{x^3}$ (p-series)

$p=3>1$, converges.

⑥ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{x}} = \sum_{n=1}^{\infty} \frac{1}{x^{\frac{1}{2}}}$ (p-series)

$p=\frac{1}{2}<1$, diverges

$\boxed{\text{new}}$ The Comparison Test

Part I: Direct Comparison

A. If $a_n < b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

B. If $a_n > b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

[Examples] Determine whether the series converges or diverges.

① $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

sidework:

$\frac{5}{5n-1} \overset{?}{>} \frac{5}{5n}$

$\frac{5}{5n-1} \overset{?}{>} \frac{1}{n}$

$5n \overset{?}{>} 5n-1$

$0 > -1$ ✓

compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent ($p \leq 1$)

Therefore, since $\frac{5}{5n-1} > \frac{1}{n}$ then $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ is divergent.

② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n+4}$

Sidework:

$$\frac{1}{n^3+n+4} < \frac{1}{n^3}$$

$n^3 < n^3+n+4$

$0 < n+4$ ✓

compare to $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent ($p>1$; p-series)

Therefore, since $\dfrac{1}{n^3+n+4} < \dfrac{1}{n^3}$ and $\sum \dfrac{1}{n^3}$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n+4}$ is convergent.

③ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Sidework:

$$\frac{1}{2^n-1} < \frac{1}{2^n}$$

$2^n-1 < 2^n$

✓ $-1 < 0$

compared to $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent ($r<1$; geometric series)

Therefore, since $\dfrac{1}{2^n-1} < \dfrac{1}{2^n}$ and $\sum \dfrac{1}{2^n}$ is convergent, then

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent.

④ $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^3}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2}{n^{\frac{1}{2}}}$ is divergent ($p<1$; p-series)

Since $\dfrac{2n^2+3n}{\sqrt{5+n^3}} > \dfrac{2}{\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ is divergent

then $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^3}}$ is divergent.

sidework:

$$\frac{2n^2+3n}{\sqrt{5+n^5}} > \frac{2n^2}{\sqrt{n^5}}$$

$$\frac{2n^2+3n}{\sqrt{5+n^5}} > \frac{2n^2}{n^{\frac{5}{2}}}$$

$$\frac{2n^2+3n}{\sqrt{5+n^5}} > \frac{2}{n^{\frac{1}{2}}}$$

$$\frac{2n^2+3n}{\sqrt{5+n^5}} > \frac{2}{\sqrt{n}}$$

## Part II: Limit Comparison

Let $\sum a_n$ and $\sum b_n$ be series with positive terms.

Then, if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $c > 0$, then both series converges

or both series diverges.

[Examples] Determine whether the series converges or diverges.

① $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$

Try DIRECT COMPARISON:

sidework:

$\frac{1}{n^2-1} \overset{?}{>} \frac{1}{n^2}$

$n^2 > n^2-1$

$0 > -1$

* Although the inequality statement is true, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (but $a_n > b_n$) so can't use direct comparison.

$\Longrightarrow a_n = \frac{1}{n^2-1}$ ; $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{2n} = \lim_{n\to\infty} 1 = 1 > 0;$$

Limit comparison can be used.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p > 1$; p-series)

Since $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by limit comparison $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ convergences.

② $\sum_{n=1}^{\infty} \frac{n^3+1}{n^4-5n-5}$

Let $a_n = \frac{n^3+1}{n^4-5n-5}$ and $b_n = \frac{n^3}{n^4} = \frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{n^3+1}{n^4-5n-5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(n^3+1)}{n^4-5n-5} = \lim_{n\to\infty} \frac{n^4+n}{n^4-5n-5}$$

$$\underset{=}{\text{L'H}} \lim_{n\to\infty} \frac{4n^3+1}{4n^3-5} \underset{=}{\text{L'H}} \lim_{n\to\infty} \frac{12n^2}{12n^2} = 1 > 0 \checkmark$$ Limit comparison can be used

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p<1$; p-series)

Since $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then by limit comparison

$\sum_{n=1}^{\infty} \frac{n^3+1}{n^4-5n-5}$ diverges.

④ $\sum_{n=1}^{\infty} \frac{2}{n^3-4}$

Let $a_n = \frac{2}{n^3-4}$ and $b_n = \frac{1}{n^3}$

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{2}{n^3-4}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2n^3}{n^3-4} = 2 > 0 \checkmark$$ limit comparison can be used.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges ($p>3$; p-series)

Since $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, then by limit comparison

$\sum_{n=1}^{\infty} \frac{2}{n^3-4}$ converges.

# SUMMARY for Convergence or Divergence

## Convergence

**1. Geometric Series:**

if $|r| < 1$ for $a_n = a(r)^n$, then $\sum a_n$ converges at $\frac{a}{1-r}$.

**2. $N^{th}$ Term Test**

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**3. p-series:**

if $p > 1$, $\sum \frac{1}{k^p}$ converges

**4. The Integral Test:**

if $\int_1^\infty f(x)\,dx$ converges, then $\sum a_k$ converges

**5. The Comparison Test**

(Direct Comparison)

if $a_n < b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(Limit Comparison)

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

## Divergence

**1. Geometric Series:**

if $|r| > 1$ for $a_n = a(r)^n$, then $\sum a_n$ diverges.

**2. $N^{th}$ Term Test:**

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

**3. p-series:**

if $p \leq 1$, $\sum \frac{1}{k^p}$ diverges

**4. The Integral Test:**

if $\int_1^\infty f(x)\,dx$ diverges, then $\sum a_k$ diverges.

**5. The Comparison Test**

(Direct Comparison)

if $a_n > b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

(Limit Comparison)

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Homework page 522 ] 1-5 (Quick Review) ; 5-8, 10, 15 (Section Review)