6.5 Alternating Series Test

$\boxed{\text{old}}$ Convergence or Divergence

[Examples] Determine whether convergent or divergent.

① $\displaystyle\sum_{n=1}^{\infty} \frac{8^{n+1}}{3^n}$ (Geometric Series)

$$\sum_{n=1}^{\infty} \frac{8^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{8^1 \cdot 8^n}{3^n} = \sum_{n=1}^{\infty} 8\left(\frac{8}{3}\right)^n$$

$r > 1$; divergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{8^{n+1}}{3^n}$ is divergent.

② $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+5}$ (N^{th} Term Test)

$$\Rightarrow \lim_{n\to\infty} \frac{n}{n+5} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0 \text{; divergent}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+5}$ is divergent.

③ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{n^2+2n}$ ((Comparison Test – Direct)

sidework:

$$\frac{1}{n^2+2n} < \frac{1}{n^2}$$

$$n^2 < n^2+2n$$

$$\checkmark\ 0 < 2n$$

$\xRightarrow{\text{compare to}} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p > 1$)

Since $\dfrac{1}{n^2+2n} < \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then $\displaystyle\sum \frac{1}{n(n+2)}$ converges.

④ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$ (Integral Test)

$$\sum_{n=1}^{\infty} \frac{1}{3n+1} = \int_1^{\infty} \frac{1}{3x+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{3x+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{u}\,du = \lim_{t\to\infty} \ln|u| \Big]_1^t$$

$u = 3x+1$
$du = 3dx$
$\frac{1}{3}du = 1dx$

$$= \lim_{t\to\infty} \ln|t| - \ln|1| = \infty; \text{ divergent}$$

$\ln|t| \to \infty$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$ is divergent.

$\boxed{\text{new}}$ Alternating Series Test

Let's consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{k}$. Express the series in a sum.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

"alternating series"

more specifically "alternating harmonic series"

Definition Alternating Series

Let $\{b_n\}$ be a sequence of positive numbers. A series formed by

$$\sum (-1)^n b_n \quad \text{or} \quad \sum (-1)^{n-1} b_n$$

is called an alternating series.

From previous example, let's graph the series using the partial sums

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$S_1 = 1$

$S_2 = .5$

$S_3 = .8\overline{33}$

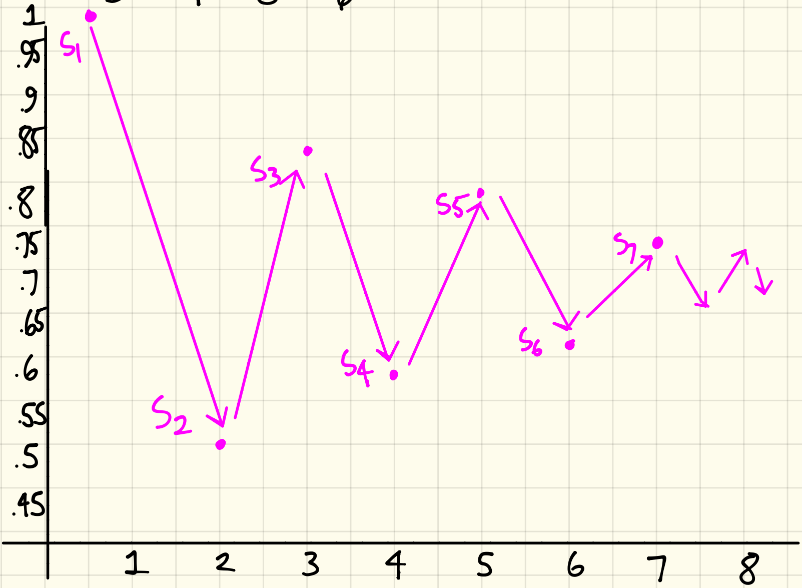
$S_4 = .58\overline{3}$

$S_5 = .78\overline{3}$

$S_6 = .61\overline{6}$

$S_7 = .759$

$\vdots$



[Alternating Series Test]

Let $\sum a_k$ be an alternating series.

If 1. $a_{k+1} < a_k$ (a_k is decreasing)

2. $\lim_{k \to \infty} a_k = 0$

Then, $\sum a_k$ is convergent.

$\boxed{\text{note:}}$ If one of the 2 conditions fails, $\sum a_k$ is divergent.

[Examples] Determine whether convergent or divergent.

① $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

1^{st} - decreasing? : $\left\{\frac{1}{2n+1}\right\} = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ ✓

2^{nd} - $\lim_{n\to\infty} a_n = 0$? $\lim_{n\to\infty} \frac{1}{2n+1} = 0.$ ✓

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ is convergent.

② $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$

1^{st} - decreasing? $\left\{\frac{3n}{4n-1}\right\} = 1, \frac{6}{7}, \frac{9}{11}, \ldots$ ✓

2^{nd} - $\lim_{n\to\infty} a_n = 0$? $\lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0.$

(Since one condition fails) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is divergent.

③ $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$

1^{st} – decreasing? $\left\{\frac{n^2}{n^3+1}\right\} = \frac{1}{2}, \frac{4}{9}, \frac{9}{28}, \ldots$ ✓

2^{nd} – $\lim_{n\to\infty} a_n = 0$? $\lim_{n\to\infty} \frac{n^2}{n^3+1} \underline{\underline{L'H}} \lim_{n\to\infty} \frac{2n}{3n^2} \underline{\underline{L'H}} \lim_{n\to\infty} \frac{2}{6n} = 0$ ✓

$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ is convergent.

Homework page 523] 18–21, supplementary worksheet.

SUMMARY for Convergence or Divergence

Convergence

1. Geometric Series:

if $|r| < 1$ for $a_n = a(r)^n$, then $\sum a_n$ converges at $\frac{a}{1-r}$.

2. N^{th} Term Test

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3. p-series:

if $p > 1$, $\sum \frac{1}{k^p}$ converges

4. The Integral Test:

if $\int_1^\infty f(x)\,dx$ converges, then $\sum a_k$ converges

5. The Comparison Test

(Direct Comparison)

if $a_n < b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(Limit Comparison)

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

6. Alternating Series Test

If a_n is decreasing & $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.

Divergence

1. Geometric Series:

if $|r| > 1$ for $a_n = a(r)^n$, then $\sum a_n$ diverges.

2. N^{th} Term Test:

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

3. p-series:

if $p \leq 1$, $\sum \frac{1}{k^p}$ diverges

4. The Integral Test:

if $\int_1^\infty f(x)\,dx$ diverges, then $\sum a_k$ diverges.

5. The Comparison Test

(Direct Comparison)

if $a_n > b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

(Limit Comparison)

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

6. Alternating Series Test

If a_n is not decreasing <u>or</u> $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.