6.6 Absolute Convergence & The Ratio and Root Tests

$\boxed{\text{Old}}$ Alternating Series Test

① $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(1+n)}$

1st - decreasing? $\left\{ \frac{1}{\ln(1+n)} \right\} = \frac{1}{\ln(2)}, \frac{1}{\ln(3)}, \frac{1}{\ln(4)}, \ldots$ ✓

2nd - $\lim_{n\to\infty} a_n = 0$? $\lim_{n\to\infty} \frac{1}{\ln(1+n)} \stackrel{\text{L'H}}{=} \lim_{n\to\infty} \frac{0}{\frac{1}{1+n}} = 0.$ ✓

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(1+n)}$ is convergent.

② $\sum_{n=1}^{\infty} (-1)^n \frac{2n+3}{3n+4}$

1st - decreasing? $\left\{ \frac{2n+3}{3n+4} \right\} = \frac{5}{7}, \frac{7}{10}, \frac{9}{13}, \ldots$ ✓

2nd - $\lim_{n\to\infty} a_n = 0$? $\lim_{n\to\infty} \frac{2n+3}{3n+4} \stackrel{\text{L'H}}{=} \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3} \neq 0.$ X

$\sum_{n=1}^{\infty} (-1)^n \frac{2n+3}{3n+4}$ is divergent.

$\boxed{\text{new} - A}$ Absolute Convergence, Conditioning Convergence, & Divergence.

Definitions: Consider $\sum a_n$.

1. Absolutely Convergent

Suppose $\sum |a_n|$ converges.

↳ Every absolutely convergent series converges (always).

That is, if $\sum |a_n|$ converges, then $\sum a_n$ converges.

2. Conditionally Convergent

Suppose $\sum |a_n|$ diverges, but $\sum a_n$ converges.

↳ The positive terms diverge but the alternating series converges.

3. Divergent

Both $\sum |a_n|$ and $\sum a_n$ diverge.

[Examples] Determine whether series absolutely converges, conditionally converges & diverges.

① $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

$\sum |a_n|$ converges or diverges?

$\sum \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum \frac{1}{n^2}$ (p-series)

$p > 1$

Since $\sum |a_n|$ converges, $\sum \left|\frac{(-1)^{n-1}}{n^2}\right|$ absolutely converges.

② $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

1st — $\sum |a_n|$ converges or diverges?

$\sum \left|\frac{(-1)^{n-1}}{n}\right| = \sum \frac{1}{n}$ (p-series)

$p \leq 1$

$\sum \left|\frac{(-1)^{n-1}}{n}\right|$ is divergent.

2nd — $\sum a_n$ converges or diverges?

$\sum \frac{(-1)^{n-1}}{n}$ (alternating series test)

$\left\{\frac{1}{n}\right\}$ decreasing? $1, \frac{1}{2}, \frac{1}{3}, \ldots$

$\lim_{n\to\infty} \frac{1}{n} = 0$? $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\sum \frac{(-1)^{n-1}}{n}$ converges.

Thus, $\sum \frac{(-1)^{n-1}}{n}$ conditionally converges.

③ $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+5}$

1st — $\sum |a_n|$ converges or diverges?

$\sum \left|(-1)^n \frac{n}{n+5}\right| = \sum \frac{n}{n+5}$ (n^{th} term test)

$\lim_{n\to\infty} \frac{n}{n+5} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0$ ✓

$\sum \left|(-1)^n \frac{n}{n+5}\right|$ diverges.

2nd — $\sum a_n$ converges or diverges?

$\sum (-1)^n \frac{n}{n+5}$ (alternating series test)

(i) $\left\{\frac{n}{n+5}\right\}$ decreasing? $\frac{1}{6}, \frac{2}{7}, \frac{3}{8}, \ldots$

(ii) $\lim_{n\to\infty} \frac{n}{n+5} = 0$? $\lim_{n\to\infty} \frac{n}{n+5} \neq 0$

Thus, $\sum (-1)^n \frac{n}{n+5}$ diverges.

$\boxed{\text{new-B}}$ The Ratio Test

Consider $\sum a_n$.

1. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is convergent (also absolutely convergent)

2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (or equal ∞), then $\sum a_n$ is divergent.

note: If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then can't say anything.

[Examples] Converges or Diverges?

1. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n}$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{\frac{n+1}{8^{n+1}}}{\frac{n}{8^n}}\right| = \lim_{n\to\infty} \frac{n+1}{8^{n+1}}\cdot\frac{8^n}{1} = \lim_{n\to\infty}\frac{n+1}{8^1\cancel{8^n}}\cdot\frac{\cancel{8^n}}{n}$$

$$= \lim_{n\to\infty}\frac{n+1}{8}\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{8}\cdot\frac{n+1}{n} = \frac{1}{8}\lim_{n\to\infty}\frac{n+1}{n}$$

$$\overset{L'H}{=} \frac{1}{8}\lim_{n\to\infty}\frac{1}{1} = \frac{1}{8} < 1$$; (absolutely) convergent.

$\sum_{n=1}^{\infty}\frac{(-1)^n}{8^n}$ converges (absolutely).

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)^3}}{\frac{3^n}{n^3}} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n} = \lim_{n\to\infty} \frac{3^1 \cdot \cancel{3^n}}{(n+1)^3} \cdot \frac{n^3}{\cancel{3^n}}$$

$$= \lim_{n\to\infty} \frac{3}{(n+1)^3} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{3}{1} \cdot \frac{n^3}{(n+1)^3} = \lim_{n\to\infty} 3 \cdot \left(\frac{n}{n+1}\right)^3$$

$$= \lim_{n\to\infty} 3 \cdot (1)^3 = 3 > 1 \text{; divergent.}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{2^1 \cdot \cancel{2^n}} \cdot \frac{\cancel{2^n}}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{1}{2}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} = (1)^2 \cdot \frac{1}{2} \text{; converges (absolutely).}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges (absolutely).

$\boxed{\text{new-C}}$ The Root Test

Consider $\sum a_n$.

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is convergent (also absolutely convergent)

2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ (or equal ∞), then $\sum a_n$ is divergent.

note: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then can't say anything.

[Examples] Converges or Diverges.

① $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} \overset{L'H}{=} \lim_{n\to\infty} \frac{0}{\frac{1}{n}} = 0 < 1$; convergent (absolutely).

$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges (absolutely).

② $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{4n} = \frac{1}{2} < 1$; convergent (absolutely).

③ $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^{1+3n}}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^1 \cdot 3^{3n}}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}} \cdot 3^3} = \lim_{n\to\infty} \frac{1}{9} \cdot \frac{n}{3^{\frac{1}{n}}}$

$= \lim_{n\to\infty} \frac{1}{9} \cdot \frac{n}{3^{\frac{1}{n}}} \to 0 = \lim_{n\to\infty} \frac{1}{9} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{9} n = \infty$; divergent.

$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ diverges.