6.7 Power Series

0|d Geometric Series

Let's recall the formula for convergence involving geometric series:

$$\sum_{n=0}^{\infty} a_1 (r)^{n-1} = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots$$

$$= \frac{a_1}{1-r} \quad \text{if } |r| < 1.$$

[Examples]

① $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{3^k}$

$$\sum_{k=0}^{\infty} \frac{2^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1-\left(\frac{2}{3}\right)} = \frac{1}{\frac{1}{3}} = 3.$$

② $\displaystyle\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^n$

$$\sum_{k=0}^{\infty} 4\left(\frac{1}{4}\right)^n = \frac{4}{1-\left(\frac{1}{4}\right)} = \frac{4}{\frac{3}{4}} = 4 \cdot \frac{4}{3} = \frac{16}{3}.$$

(3) $\displaystyle\sum_{k=0}^{\infty} \frac{3^{k+1}-2^k}{5^k}$

$$\sum_{k=0}^{\infty} \frac{3^{k+1}-2^k}{5^k} = \sum_{k=0}^{\infty} \frac{3^{k+1}}{5^k} - \sum_{k=0}^{\infty} \frac{2^k}{5^k}$$

$$= \sum_{k=0}^{\infty} \frac{3^1 3^k}{5^k} - \sum_{k=0}^{\infty} \frac{2^k}{5^k}$$

$$= \sum_{k=0}^{\infty} 3\left(\frac{3}{5}\right)^k - \sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k$$

$$= \frac{3}{1-\left(\frac{3}{5}\right)} - \frac{1}{1-\left(\frac{2}{5}\right)}$$

$$= \frac{3}{\frac{2}{5}} - \frac{1}{\frac{3}{5}} = \frac{35}{6}$$

$\boxed{\text{new}}$ Power Series

Let's recall the geometric series: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

- series converges: $|x| < 1$
- series diverges. otherwise

$\boxed{\text{Definition}}$ Power Series

Let's consider the function $f(x)$.

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{1-x} \text{ (closed form) } ; \ |x| < 1.$$

Goal To manipulate the power series function to be able to use the formula to determine the closed form and interval of convergence.

[Examples] Find the closed formula & interval of convergence for each power series.

① $\sum_{n=0}^{\infty} (-1)^k x^k$

$$\sum (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots$$

$$= \sum (-1x)^k = \frac{1}{1-(-1x)} = \frac{1}{1+x}$$

$$|-x| < 1$$

$$|-1|\,|x| < 1$$

$$-1 < x < 1$$

② $\sum_{n=0}^{\infty} 2^n x^{n+2}$

$$= \sum 2^n x^n x^2 = x^2 \sum (2x)^n = \frac{x^2}{1-(2x)}$$

$$\Rightarrow |2x| < 1$$

$$|x| < \frac{1}{2}$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

③ $\sum_{n=0}^{\infty} (-1)^k x^{2k}$

$$= \sum (-1)^k (x^2)^k = \sum (-1x^2)^k = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

$$\Rightarrow |-x^2| < 1$$

$$|x| < 1$$

$$-1 < x < 1.$$

④ $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{3^k}$

$$= \sum \frac{x^{2k} x^1}{3^k} = x \sum \frac{(x^2)^k}{3^k} = x \sum \left(\frac{x^2}{3}\right)^k = \frac{x}{1-\left(\frac{x^2}{3}\right)}$$

$$\Rightarrow \left|\frac{x^2}{3}\right| < 1$$

$$|x|^2 < 3$$

$$|x| < \sqrt{3}$$

$$-\sqrt{3} < x < \sqrt{3}$$

[Radius of Convergence]

General Definition of Power Series

A power series is a series of the form — centered at $x=0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$, where c_n is the coefficents

A power series is a series of the form — centered at $x=a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots,$$ where $a \in \mathbb{R}$.

(Recall that a power series may converge for some values of x and diverge for other values of x)

Theorem

For any given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

1. The series converges only when $x=a$.
2. The series converge for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

note The number R represents the Radius of Convergence of the power series. By convention, the radius of convergence is $R=0$ [for case 1] and $R=\infty$ [for case 2].

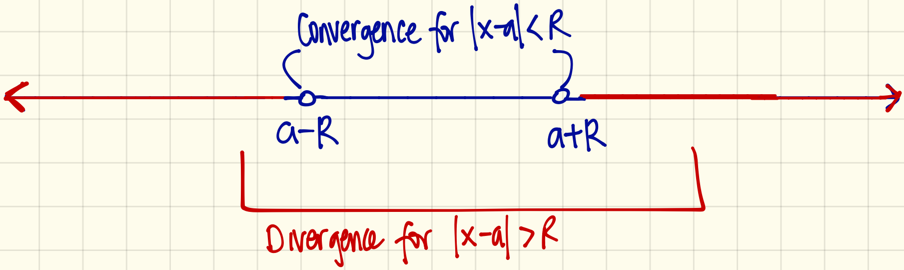
What about case 3?

Let's recall that the radius of convergence for power series is

$$|x-a| < R$$

$$\Rightarrow -R < x-a < R$$

$$\Rightarrow a-R < x < a+R$$



$\boxed{\text{note:}}$ Using convergence test like root & ratio tests can help.

[Examples] Find the interval & radius of convergence.

① $\sum_{n=0}^{\infty} n! \cdot x^n$ (ratio test)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n! \quad x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{n!}}\,\frac{\cancel{x^{n+1}}^{\,x}}{\cancel{x^n}}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

series converges when $x=0$;

interval of convergence $\{0\}$
radius of convergence $R=1$.

② $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ (ratio test)

$$\lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{(n+1)}}{\frac{(x-3)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{\cancelto{(x-3)^1}{(x-3)^{n+1}}}{(n+1)} \cdot \frac{n}{\cancel{(x-3)^n}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) |x-3|$$

$= |x-3| < 1$

Radius of Convergence

$|x-3| < 1$

Radius of Convergence: 1

Interval of Convergence

$|x-3| < 1$

$-1 < x-3 < 1$

$2 < x \quad 4$

Check endpts for convergence:

$x=2$

$\sum \frac{(x-3)^n}{n} = \sum \frac{(2-3)^n}{n} = \sum \frac{(-1)^n}{n}$ (Alternating Series Test)

1) decreasing ✓

2) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

∴ Convergent

$x=4$

$\sum \frac{(x-3)^n}{n} = \sum \frac{(4-3)^n}{n} = \sum \frac{1}{n}$ (p-series) $p=1$

∴ divergent.

$2 \le x < 4$

$[2,4)$

② $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ (Ratio Test)

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}}{\frac{n(x+2)^n}{3^{n+1}}}\right| = \lim_{n\to\infty} \frac{(n+1)\cancel{(x+2)^{n+1}}^{(x+2)}}{\cancel{3^{n+2}}_{(3)}} \cdot \frac{\cancel{3^{n+1}}}{n\cancel{(x+2)^n}} = \lim_{n\to\infty} \frac{(n+1)}{n} \cdot \frac{|x+2|}{3}$$

$$= \frac{|x+2|}{3} < 1$$

Radius of Convergence

$$\frac{|x+2|}{3} < 1$$

$$|x+2| < 3$$

Radius of Convergence = 3

Interval of Convergence

$$|x+2| < 3$$

$$-3 < x+2 < 3$$

$$-5 < x < 1$$

check endpoints:

$x=-5$

$$\sum_{n=0}^{\infty} \frac{n(-5+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^1 3^n} = \frac{1}{3}\sum n\left(\frac{-3}{3}\right)^n = \frac{1}{3}\sum n(-1)^n$$

$= \infty$; divergent.

$x=1$

$$\sum_{n=0}^{\infty} \frac{n(1+2)^n}{3^{n+1}} = \frac{1}{3}\sum n = \infty$$; divergent.