6.8 Representations of Functions as Power Series

$\boxed{0|d}$ Power Series

[Examples] Find the radius & interval of convergence.

① $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ (Ratio Test)

$$\Rightarrow \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{1}{n+1}\cdot|x| = 0\cdot|x| = 0<1$$

(absolutely convergence)

Radius of Convergence: $\mathbb{R}$

Interval of Convergence: $(-\infty,\infty)$

② $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n$ (Ratio Test)

$$\Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)-1}}{(n+1)}x^n}{\frac{(-1)^{n-1}}{n}x^n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\cdot|x| = 1\cdot|x|<1$$

Interval of Convergence

$|x|<1$

$-1<x<1$

Check endpts:

$x=-1$ $\sum\frac{(-1)^{n-1}(-1)^n}{n} = \sum\frac{-1}{n}$ $\therefore$ divergent (p-series)

$x=1$ $\sum\frac{(-1)^{n-1}}{n}$ (Alternating Series Test) $\therefore$ convergent (conditionally)

So, $-1<x\leq 1$ or $(-1,1]$.

Radius of Convergence

$|x|<1$

R of Convergence: 1

[new] Representations of Functions using Power Series

Let's consider the function $f(x)$.

$$f(x)=\sum_{k=0}^{\infty} x^k = 1+x+x^2+x^3+\cdots$$
$$= \frac{1}{1-x} \text{ (closed form) }; \quad |x|<1.$$

Goal: To manipulate the closed form to create the power series function.

[Examples] Find the power series function.

① $\dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1x^2)^n=\sum_{n=0}^{\infty}(-1)^n(x^2)^n$$

② $\dfrac{1}{2+x}$

$$\frac{1}{2+x}=\frac{1}{2\left(1+\frac{x}{2}\right)}=\frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]}=\frac{1}{2}\cdot\frac{1}{1-\left(-\frac{x}{2}\right)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-1x}{2}\right)^n$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\cdot 2}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n.$$

③ $\dfrac{x^3}{x+2}$

$$\frac{x^3}{x-4} = \frac{x^3}{4-x} = x^3 \cdot \frac{1}{4(1-\frac{x}{4})} = \frac{x^3}{4(1-\frac{x}{4})} = \frac{x^3}{4} \cdot \frac{1}{(1-\frac{x}{4})} = \frac{x^3}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^3}{4}\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty} \frac{x^3 \cdot x^n}{4 \cdot 4^n} = \sum_{n=0}^{\infty} \frac{x^{3+n}}{4^{n+1}}.$$

[Differentiation & Integration in Power Series]

Let's consider the sum of the series: $\dfrac{1}{1-x}$.

$$\Rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x|<1.$$

Differentiate the series:

LHS: $\dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{(1-x)(0)-(1)(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$

Find the power series using the differentiated sum of series:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty} nx^{n-1}$$

Now let's integrate of the considered function:

LHS: $\int \frac{1}{1-x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + c = -\ln(1-x)$

$u = 1-x$
$du = -1dx$

Find the power series using the integrated sum of series:

$$-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

Theorem

Let $f(x) = \sum c_k (x - x_0)^k$ be a power series with a nonzero radius of convergence. Then,

- $f'(x) = \sum c_k\, k (x - x_0)^{k-1}$ for $|x - x_0| < R$

and

- $\int f(x)\,dx = \sum \frac{c_k}{k+1}(x - x_0)^{k+1} + C$ for $|x - x_0| < R$.

$\boxed{\text{note:}}$ The radius of convergence remains the same when power series is differentiated or integrated. However, the interval of convergence may not remain the same.

[Examples] Find the power series representations.

① $f(x) = \tan^{-1} x$

note: $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}\, dx$

$$\tan^{-1} x = \int \frac{1}{1+x^2} dx = \int \frac{1}{1-(-x^2)} dx = \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Alternative way using theorem:

$$\tan^{-1} x = \int \sum_{n=0}^{\infty} (-x^2)^n dx = \int \sum (-1)^n (x)^{2n} dx = \sum (-1)^n \frac{(x)^{2n+1}}{2n+1} + C.$$

② $f(x) = \ln(1+x)$

note: $\frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$

$$\ln(1+x) = \int \frac{1}{1+x} dx = \int \frac{1}{1-(-x)} dx = \int \sum (-x)^n dx = \sum (-1)^n \frac{x^{n+1}}{n+1} + C$$