6.9 Taylor Series & Maclaurin Series

$\boxed{\text{Old}}$ → we were able to find power series for a certain restricted class of functions.

$\boxed{\text{now}}$ → we are finding functions for powerseries for more general problems.

[Maclaurin Series] Taylor Series centered at 0.

Let's consider a function $f(x)$ defined by a power series about 0 with a nonzero radius of convergence.

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots \qquad f(0) = c_0$$
$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots \qquad f'(0) = c_1$$
$$f''(x) = 2c_2 + 3\cdot 2c_3x + 4\cdot 3c_4x^2 + \cdots \qquad f''(0) = 2c_2$$
$$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4x + 5\cdot 4\cdot 3c_5x^2 + \cdots \qquad f'''(0) = 3\cdot 2c_3$$
$$f^{(4)}(x) = 4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5x + \cdots \qquad f^{(4)}(0) = 4\cdot 3\cdot 2c_4$$
$$\vdots \qquad \vdots$$
$$f^{(k)}(x) = (k!)c_k + (k+1)!\,c_{k+1}x + \cdots \qquad f^{(k)}(0) = (k!)\,c_k$$

What is the coefficient, c_k?

$$f^{(k)}(0) = (k!)c_k$$

$$c_k = \frac{f^{(k)}(0)}{(k)!}$$

Theorem (Maclaurin Series)

So, if $f(x)$ can be represented by a power series on $(-R,R)$, then the power series may be written as

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

$$=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)x^3}{3!}+\cdots$$

[Examples] Find the Maclurin Series of the function $f(x)=e^x$ & its radius of convergence.

1

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{(n)!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}+x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

$f(x)=e^x \qquad f(0)=1$

$f'(x)=e^x \qquad f'(0)=1$

$f''(x)=e^x \qquad f''(0)=1$

$f'''(x)=e^x \qquad f'''(0)=1$

$\vdots \qquad \vdots$

$$=1+1x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

(use ratio test to determine radius of convergence)

$$\Rightarrow \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{\,x}}{\cancel{(n+1)!}_{\,n+1}}\cdot\frac{\cancel{n!}}{\cancel{x^n}}\right|=\lim_{n\to\infty}\frac{1}{n+1}\cdot|x|=0<1 \therefore \text{absolutely convergent}$$

Radius of Convergence: ∞

② $f(x) = \sin x$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{(n)!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} + x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

$$f(x) = \sin x \longrightarrow f(0) = 0$$
$$f'(x) = \cos x \longrightarrow f'(0) = 1$$
$$f''(x) = -\sin x \longrightarrow f''(0) = 0$$
$$f'''(x) = -\cos x \longrightarrow f'''(0) = -1$$
$$f^{(4)}(x) = \sin x \longrightarrow f^{(4)}(0) = 1$$
$$\vdots \qquad\qquad \vdots$$

$$\sin x = 0 + 1x - \frac{0x^2}{2!} - \frac{1}{3!} x^3 + \frac{0x^4}{4!} + \frac{1}{5!} x^5 + \frac{0}{6!} x^6 - \frac{1}{7!} x^7 + \cdots$$

$$\sin x = 1 - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

[Taylor Series centered at x_0],

In the same way as before (as centered at $x=0$), we find that

$$c_K = \frac{f^{(K)}(x_0)}{K!}$$

Therefore, the Taylor Series expansion of f centered at x_0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

[Example 1] Find the Taylor series expansion of $\sin x$ centered a $\frac{\pi}{4}$.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

$$f(x)=\sin x \longrightarrow f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f'(x)=\cos x \longrightarrow f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f''(x)=-\sin x \longrightarrow f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$f'''(x)=-\cos x \longrightarrow f'''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$f^{(4)}(x)=\sin x \longrightarrow f^{(4)}\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$\sin x=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2}\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}-\frac{\sqrt{2}}{2}\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}+\cdots$$

$$=\frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{\frac{K}{2}}}{K!}\left(x-\frac{\pi}{4}\right)^K.$$