### 7.2 Tangents \& Areas

Old Parametric Equations
[Examples] Find the cartesian equation of the curve.

$$
\begin{aligned}
& \text { (1) } x=2 t+4, y=t-1 \\
& \begin{aligned}
\Rightarrow \begin{array}{l}
y=t-1 \\
t=y+1
\end{array} \\
x=2 y+6 .
\end{aligned} \quad \begin{array}{r}
x=2(y+1)+4 \\
\\
\\
\\
\\
=2 y+2+4
\end{array} \\
& 2 y+6 .
\end{aligned}
$$

(2)

$$
\begin{aligned}
& \text { (2) } \begin{aligned}
x= & \sqrt{t}, y=1-t \\
\Rightarrow & x=\sqrt{t} x \geq 0 \\
& t=x^{2}
\end{aligned} \quad \Rightarrow \begin{array}{l}
y=1-\left(x^{2}\right) \\
y=1-x^{2} .
\end{array} \\
& y=1-x^{2}, x \geq 0
\end{aligned}
$$

(3)

$$
\begin{aligned}
& x=e^{t}, y=e^{-t} \\
& \Rightarrow x=e^{t} \\
& \ln x=t x>0 \\
& \Rightarrow y=e^{-t} \\
& \begin{aligned}
y & =e^{-(\ln x)} \\
= & =x^{-1}
\end{aligned} \\
& =e^{\ln x^{-1}} \\
& =x^{-1} \\
& =\frac{1}{x} \\
& y=\frac{1}{x} \quad x>0 .
\end{aligned}
$$

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[Examples] Describe the motion of a particle with position $(x, y)$ as varies in the given interval.
(4)

$$
\begin{aligned}
& x=2 \sin t, y=3 \cos t, 0 \leq t \leq 2 t \\
& \Rightarrow x=2 \sin t \\
& \Rightarrow y=3 \cos t \\
& \frac{x}{2}=\sin t \\
& \left(\frac{x}{2}\right)^{2}=(\sin t)^{2} \\
& \frac{y}{3}=\cos t \\
& \left(\frac{y}{3}\right)^{2}=(\cos t)^{2} \\
& \frac{x^{2}}{4}=\sin ^{2} t \\
& \frac{y^{2}}{9}=\cos ^{2} t
\end{aligned}
$$

$$
\begin{aligned}
& \sin ^{2} t+\cos ^{2} t=1 \\
& \frac{x^{2}}{4}+\frac{y^{2}}{9}=1
\end{aligned}
$$



| $t$ | $x$ | $y$ |
| :---: | :---: | :---: |
| 0 | 0 | 3 |
| $\frac{\pi}{6}$ | 1 | $\frac{3 \sqrt{3}}{2}$ |
| $\frac{\pi}{4}$ | $\sqrt{2}$ | $\frac{3 \sqrt{2}}{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

$\therefore$ moving clockwise around an ellipse.

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new Tangents \& Areas
[Tangents]
Let's consider the parametric equations: $x=f(t), y=g(t)$.
When we eliminate the parameter to create one equation, we get

$$
\begin{gathered}
y=F(x) \\
g(x)=F(f(t))
\end{gathered}
$$

Now, let's differentiate this new parametric equations: (need chain rule to differentiate)

$$
\begin{aligned}
g(t) & =F(f(t)) \\
\frac{d}{d t} g(t) & =\frac{d}{d t} F(f(t)) \\
g^{\prime}(t) & =F^{\prime}(f(t)) \cdot f^{\prime}(t) \\
& =F^{\prime}(x) f^{\prime}(t)
\end{aligned}
$$

Now, let's solve for $F^{\prime}(x)$, assuming that $\left(f^{\prime}(t) \neq 0\right)$ :

$$
\begin{aligned}
& g^{\prime}(t)=F^{\prime}(x) f^{\prime}(t) \\
& F^{\prime}(x)=\frac{g^{\prime}(t)}{f^{\prime}(t)}
\end{aligned}
$$

This can be rewritten as $F^{\prime}(x)=\frac{\frac{d y}{d t}}{\frac{d x}{d t}} \quad$ (provided that $\left.\frac{d x}{d t} \neq 0\right)$.
[Examples] Find the slope of each point given.

$$
\begin{aligned}
& \text { (1) } \left.x=\sqrt{t}, y=\frac{1}{4}\left(t^{2}-4\right) \text { at } t=4 . \quad\left(\frac{d y}{d x}\right]_{t=4}\right) \\
& x=t^{1 / 2}, y=\frac{1}{4} t^{2}-1 \\
& \frac{d y}{d x}=\frac{\frac{d y}{d t}}{\frac{d x}{d t}}=\frac{\frac{1}{4}(2) t^{1}}{\frac{1}{2} t^{-\frac{1}{2}}}=\frac{\frac{1}{2} t}{\frac{1}{2} t^{-\frac{1}{2}}}=t^{3 / 2}=\sqrt{t^{3}} \\
& \left.\frac{d y}{d x}\right|_{t=4}=\sqrt{(4)^{3}}=\sqrt{64}=8 .
\end{aligned}
$$

(2) $x=\tan t, y=\sin t$ at $t=\frac{\pi}{4}$

$$
\begin{aligned}
& \frac{d y}{d x}=\frac{\frac{d y}{d t}}{\frac{d x}{d t}}=\frac{\cos t}{\sec ^{2} t}=\frac{\cos t}{\frac{1}{\cos ^{2} t}}=\cos ^{3} t=(\cos t)^{3} \\
& \left.\frac{d y}{d x}\right]_{t=\frac{\pi}{4}}=\left[\cos \left(\frac{\pi}{4}\right)\right]^{3}=\left(\frac{\sqrt{2}}{2}\right)^{3}=\frac{2 \sqrt{2}}{8}=\frac{\sqrt{2}}{4} .
\end{aligned}
$$

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Now, let's revisit the previous considered example with the goal of finding $\frac{d^{2} y}{d x^{2}}$

$$
\frac{d^{2} y}{d x^{2}}=\frac{d}{d x}\left(\frac{d y}{d x}\right)=\frac{\frac{d}{d t} \cdot \frac{d x}{d x}}{\frac{d x}{d t}}
$$

[Examples] Find $\frac{d^{2} y}{d x}$ from each parametric equation.
(1) $x=\tan t, y=\sin t$.

Remember: $\frac{d y}{d t}=\cos t, \frac{d x}{d t}=\sec ^{2} t$ and $\frac{d y}{d x}=\cos ^{3} t$.

$$
\begin{aligned}
\frac{d^{2} y}{d x^{2}} & =\frac{\frac{d}{d t}\left(\frac{d y}{d x}\right)}{\frac{d x}{d t}}=\frac{\frac{d}{d t}\left(\cos ^{3} t\right)}{\sec ^{2} t}=\frac{3 \cos ^{2} t \cdot(-\sin t)}{\sec ^{2} t}=\frac{-3 \cos ^{2} t \sin t}{\sec ^{2} t} \\
& =-3 \sin t \cos ^{4} t .
\end{aligned}
$$

(2) $x=\sqrt{t}, y=\frac{1}{4}\left(t^{2}-4\right), t \geq 0$

Remember: $\frac{d y}{d t}=\frac{1}{2} t, \frac{d x}{d t}=\frac{1}{2} t^{-\frac{1}{2}}, \frac{d y}{d x}=t^{\frac{3}{2}}$

$$
\frac{d^{2} y}{d x^{2}}=\frac{\frac{d}{d t}\left(\frac{d y}{d x}\right)}{\frac{d x}{d t}}=\frac{\frac{d}{d t}\left(t^{\frac{3}{2}}\right)}{\frac{1}{2} t^{-\frac{1}{2}}}=\frac{\frac{3}{2} t^{\frac{1}{2}}}{\frac{1}{2} t^{-\frac{1}{2}}}=3 t
$$

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[Areas]
Let's consider the parametric curve $y=F(x)$. Recall that $x=f(t)$ and $y=g(t)$. Suppose we want to find the area between $a$ and $b$.

$$
\begin{aligned}
& y=F(x) \\
& \text { Area } \left.=\int_{a}^{b} F(x) d x=\int_{a}^{b} y d x=\int_{\alpha}^{b} g(t) f^{\prime}(t) d t . \quad \text { (since } y=g(t) \text { and } \frac{d x}{d t}=f^{\prime}(t)\right)
\end{aligned}
$$

[Example] Find the area under the arch of a cycloid with the parametric equations $x=r(1-\sin \theta)$ and $y=r(1-\cos \theta)$.

$$
\begin{aligned}
\text { Area }=\int_{0}^{2 \pi} y d x & =\int_{0}^{2 \pi} r(1-\cos \theta) r(1-\cos \theta) d \theta \\
& =r^{2 \pi}(1-\cos \theta)(1-\cos \theta) d \theta \\
& =r^{2} \int_{0}^{2 \pi} 1-2 \cos \theta+\cos ^{2} \theta d \theta \\
& =r^{2} \int_{0}^{2 \pi} 1-2 \cos \theta+\frac{1}{2}(1+\cos 2 \theta) d \theta \\
& =r^{2} \int_{0}^{2 \pi} 1-2 \cos \theta+\frac{1}{2}+\frac{1}{2} \cos 2 \theta d \theta \\
& =r^{2} \int_{0}^{2 \pi} \frac{3}{2}-2 \cos \theta+\frac{1}{2} \cos 2 \theta d \theta \\
& =r^{2}\left[\frac{3}{2} \theta-2 \sin \theta+\frac{1}{4} \sin 2 \theta\right]_{0}^{2 \pi} \\
& =r^{2}[3 \pi-0] \\
& =3 \pi r^{2}
\end{aligned}
$$

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