7.2 Tangents & Areas

$\boxed{\text{Old}}$ Parametric Equations

[Examples] Find the cartesian equation of the curve.

① $x = 2t+4, \quad y = t-1$

$\Rightarrow y = t-1$
$t = y+1$

$\Rightarrow x = 2(y+1)+4$
$= 2y+2+4$
$= 2y+6.$

$x = 2y+6.$

② $x = \sqrt{t}, \quad y = 1-t$

$\Rightarrow x = \sqrt{t} \quad x \geq 0$
$t = x^2$

$\Rightarrow y = 1-(x^2)$
$y = 1-x^2.$

$y = 1-x^2, \quad x \geq 0$

③ $x = e^t, \quad y = e^{-t}$

$\Rightarrow x = e^t$
$\ln x = t \quad x > 0$

$\Rightarrow y = e^{-t}$
$= e^{-(\ln x)}$
$= e^{\ln x^{-1}}$
$= x^{-1}$
$= \dfrac{1}{x}$

$y = \dfrac{1}{x} \quad x > 0.$

[Examples] Describe the motion of a particle with position (x,y) as t varies in the given interval.

④ $x = 2\sin t$, $y = 3\cos t$, $0 \le t \le 2t$

$\Rightarrow$ $x = 2\sin t$

$\frac{x}{2} = \sin t$

$\left(\frac{x}{2}\right)^2 = (\sin t)^2$

$\frac{x^2}{4} = \sin^2 t$

$\Rightarrow$ $y = 3\cos t$

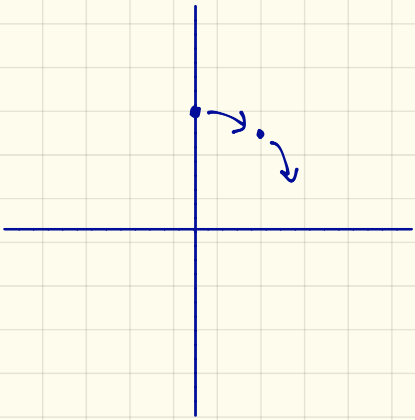
$\frac{y}{3} = \cos t$

$\left(\frac{y}{3}\right)^2 = (\cos t)^2$

$\frac{y^2}{9} = \cos^2 t$

$\sin^2 t + \cos^2 t = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$



t	x	y
0	0	3
$\frac{\pi}{6}$	1	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\sqrt{2}$	$\frac{3\sqrt{2}}{2}$
⋮	⋮	⋮

$\therefore$ moving clockwise around an ellipse.

$\boxed{\text{new}}$ Tangents & Areas

[Tangents]

Let's consider the parametric equations: $x=f(t)$, $y=g(t)$.

When we eliminate the parameter to create one equation, we get

$$y=F(x)$$
$$g(x)=F(f(t)).$$

Now, let's differentiate this new parametric equations:
(need chain rule to differentiate)

$$g(t)=F(f(t))$$
$$\frac{d}{dt}\,g(t)=\frac{d}{dt}\,F(f(t))$$

$$g'(t)=F'(f(t))\cdot f'(t)$$
$$=F'(x)f'(t)$$

Now, let's solve for $F'(x)$, assuming that $(f'(t)\neq 0)$:

$$g'(t)=F'(x)f'(t)$$

$$F'(x)=\frac{g'(t)}{f'(t)}.$$

This can be rewritten as $F'(x)=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ (provided that $\frac{dx}{dt}\neq 0$).

[Examples] Find the slope of each point given.

① $x = \sqrt{t}$, $y = \frac{1}{4}(t^2 - 4)$ at $t = 4$. $\left(\left.\frac{dy}{dx}\right]_{t=4} \right)$

$x = t^{1/2}$, $y = \frac{1}{4}t^2 - 1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{4}(2)t^1}{\frac{1}{2}t^{-\frac{1}{2}}} = \frac{\frac{1}{2}t}{\frac{1}{2}t^{-\frac{1}{2}}} = t^{3/2} = \sqrt{t^3}$$

$$\left.\frac{dy}{dx}\right|_{t=4} = \sqrt{(4)^3} = \sqrt{64} = 8.$$

② $x = \tan t$, $y = \sin t$ at $t = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{\sec^2 t} = \frac{\cos t}{\frac{1}{\cos^2 t}} = \cos^3 t = (\cos t)^3$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \left[\cos\left(\frac{\pi}{4}\right)\right]^3 = \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}.$$

Now, let's revisit the previous considered example with the goal of finding $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt} \cdot \frac{dx}{dx}}{\frac{dx}{dt}}$$

[Example1] Find $\frac{d^2y}{dx}$ from each parametric equation.

① $x = \tan t$, $y = \sin t$.

Remember: $\frac{dy}{dt} = \cos t$, $\frac{dx}{dt} = \sec^2 t$ and $\frac{dy}{dx} = \cos^3 t$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\cos^3 t)}{\sec^2 t} = \frac{3\cos^2 t \cdot (-\sin t)}{\sec^2 t} = \frac{-3\cos^2 t \sin t}{\sec^2 t}$$

$$= -3\sin t \cos^4 t.$$

② $x = \sqrt{t}$, $y = \frac{1}{4}(t^2 - 4)$, $t \geq 0$

Remember: $\frac{dy}{dt} = \frac{1}{2}t$, $\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$, $\frac{dy}{dx} = t^{\frac{3}{2}}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(t^{\frac{3}{2}}\right)}{\frac{1}{2}t^{-\frac{1}{2}}} = \frac{\frac{3}{2}t^{\frac{1}{2}}}{\frac{1}{2}t^{-\frac{1}{2}}} = 3t.$$

[Areas]

Let's consider the parametric curve $y = F(x)$. Recall that $x = f(t)$ and $y = g(t)$. Suppose we want to find the area between a and b.

$$y = F(x)$$

$$\text{Area} = \int_a^b F(x)\,dx = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt.$$ (Since $y = g(t)$ and $\frac{dx}{dt} = f'(t)$)

[Example 1] Find the area under the arch of a cycloid with the parametric equations $x = r(1 - \sin\theta)$ and $y = r(1 - \cos\theta)$.

$$\text{Area} = \int_0^{2\pi} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta$$

$$= r^2\int_0^{2\pi} (1-\cos\theta)(1-\cos\theta)\,d\theta$$

$$= r^2\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta$$

$$= r^2\int_0^{2\pi} 1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= r^2\int_0^{2\pi} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta$$

$$= r^2\int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\,d\theta$$

$$= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= r^2\left[3\pi - 0\right]$$

$$= 3\pi r^2$$