7.2 Substitution Rule

Using Trigonometric Identities

Standards:

MC11

MC11d

$\boxed{\text{Old}}$ Substitution Rule for Integration

Recall: using u-substitution technique to compute integrals involving compositions of functions.

Example Integrate.

$$\int x^2\sqrt{5+2x^3}\,dx = \frac{1}{6}\int \sqrt{u}\,du = \int \frac{1}{6}(u)^{1/2}\,du = \frac{1}{6}\left[\frac{u^{3/2}}{3/2}\right]+C$$

$u = 5+2x^3$
$du = 6x^2 \Rightarrow \frac{1}{6}du = x^2\,dx$

$$= \frac{1}{6}\left[\frac{2}{3}u^{3/2}\right]+C = \frac{1}{9}(5+2x^3)^{3/2}+C = \frac{1}{9}\sqrt{(5+2x^3)^3}.$$

$\boxed{\text{new}}$ Trig Substitution

With integrating trig functions, the basic idea is to:

1. rewritting Integral functions using trig identities.
2. use u-substitution technique

Case 1 - using reciprocal formulas

$\sin x = \frac{1}{\csc x}$ $\cos x = \frac{1}{\sec x}$ $\tan x = \frac{1}{\cot x}$ $\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\csc x}$ $\cot x = \frac{1}{\tan x}$

$\tan x = \frac{\sin x}{\cos x}$ $\cot x = \frac{\cos x}{\sin x}$

[Example] Integrate.

1. $\displaystyle\int \frac{dx}{\cos^2 x} = \int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x + C.$

② $\int \cot 7x \, dx = \int \frac{\cos 7x}{\sin 7x} dx = \frac{1}{7}\int \frac{1}{u} du = \frac{1}{7}\int u^{-1} du =$

$u = \sin 7x$
$du = \cos 7x \cdot 7 \, dx$
$du = 7\cos 7x \, dx \Rightarrow \frac{1}{7} du = \cos 7x$

$= \frac{1}{7}\left[\ln|u|\right] + C = \frac{1}{7}\ln|\sin 7x| + C.$

Case2: $\int \cos^k x \sin^l x \, dx$

when k & l are positive integers & at least one of k or l is odd.

*useful technique: save the odd factor & use $\sin^2 x + \cos^2 x = 1$.

$\boxed{\text{Examples.}}$

① $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1-\cos^2 x)\cos^2 x \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx \Rightarrow -du = \sin x \, dx$

$= -\int (1-u^2) u^2 du = -\int u^2 - u^4 du = -\left[\frac{u^3}{3} - \frac{u^5}{5} + C\right] = -\frac{u^3}{3} + \frac{u^5}{5} - C$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} - C.$

② $\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1-\sin^2 x)\cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$= \int 1 - u^2 du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$

Case 3: $\int \cos^k x \sin^l x \, dx$

when k & l are positive integers & are both even.

$\cos^2 x = \frac{1}{2}(1+\cos(2x))$ $\quad \sin(2x) = 2\sin x \cos x$

$\sin^2 x = \frac{1}{2}(1-\cos(2x))$

[Examples]

① $\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1-\cos(2x)) \cdot \frac{1}{2}(1+\cos(2x)) \, dx$

$= \frac{1}{4}\int (1-\cos(2x))(1+\cos 2x) \, dx = \frac{1}{4}\int 1 - \cos 2x + \cos 2x - \cos^2 2x \, dx$

$= \frac{1}{4}\int 1 - \cos^2 2x \, dx = \frac{1}{4}\int 1 - \left[\frac{1}{2}(1+\cos 4x)\right] dx$

$= \frac{1}{4}\int 1 - \frac{1}{2} - \frac{1}{2}\cos(4x) \, dx = \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x \, dx$

$= \frac{1}{4}\int \frac{1}{2} dx - \frac{1}{4}\cdot\frac{1}{2}\int \cos 4x \, dx$ $\quad = \frac{1}{4}\int \frac{1}{2} dx - \frac{1}{32}\int \cos u \, dx$

$u = 4x$

$du = 4\,dx \Rightarrow \frac{1}{4}du = 1\,dx$

$= \frac{1}{4}\left[\frac{1}{2}x\right] - \frac{1}{32}[\sin u] + C = \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$

Case 4: $\int \tan^k x \sec^l x \, dx$

when k & l are positive integers & K is even .

Save $\sec^2 x$ factor, use $\sec^2 x = 1 + \tan^2 x$ & $u = \tan x$

[Examples]

① $\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x (1+\tan^2 x) \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x dx$

$= \int u^2 (1+u^2) \, du = \int u^2 + u^4 \, du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$

② $\int \sec^4 x \, du = \int \sec^2 x \sec^2 x \, dx = \int (1+\tan^2 x)(\sec^2 x) \, dx$

$= \int \sec^2 x + \tan^2 x \sec^2 x \, dx = \int \sec^2 x + \int \tan^2 x \sec^2 x$

$u = \tan x$

$du = \sec^2 x$

$= \int \sec^2 x + \int u^2 du = \tan x + \frac{u^3}{3} + C$

$= \tan x + \frac{\tan^3 x}{3} + C$

Case5: $\int \tan^k x \sec^l x\, dx$

when k & l are positive integers & l is odd

save $\tan x \sec x$ factor, use $\tan^2 x = \sec^2 x - 1$ & $u = \sec x$

[Examples]

1. $\int \tan^5 x \sec^3 x\, dx = \int \tan^4 x \sec^2 x \tan x \sec x\, dx$

$= \int (\sec^2 x - 1)^2 (\sec^2 x) \tan x \sec x\, dx = \int (u^2-1)^2 u^2\, du$

$u = \sec x$

$du = \sec x \tan x\, dx$

$= \int (u^4 - 2u^2 + 1) u^2\, du = \int u^6 - 2u^4 + u^2\, du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$

$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$