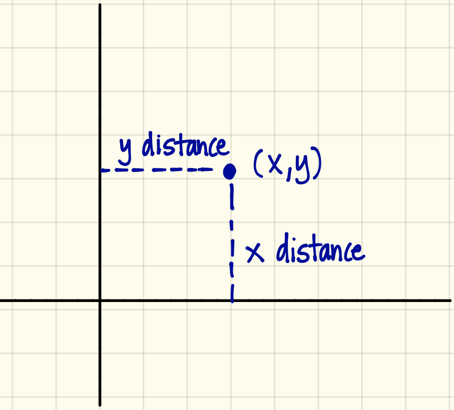
7.4 Introduction to Polar Coordinates

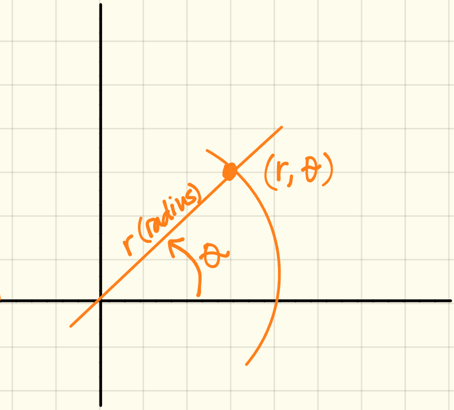
New Introduction to Polar Coordinates

[Polar Versus Rectangular Coordinates]

Let's consider a graph with an arbitrary point in the coordinate plane.

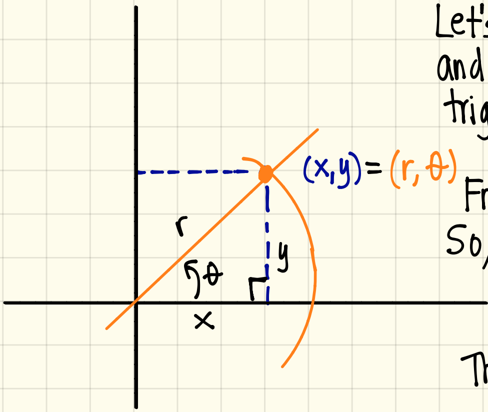


Rectangular Coordinates - measures the horizontal & vertical distances (and direction) from the axes.



Polar Coordinates - measures the distances (and direction) from the origin (radius) & the circle

Conclusion Rectangular Coordinates deal with horizontal & vertical distances whereas polar coordinates deal with diagonal & circular distances.



Let's consider the intersection of rectangular and polar coordinates. Also, let's recall that in trigonometry $x = \cos\theta$ and $y = \sin\theta$.

From the right triangle:

So, $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Also, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{y}{r}}{\frac{x}{r}} = \dfrac{y}{x}$

$\boxed{\text{Conversion Equations}}$

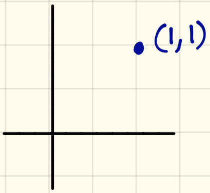
$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \dfrac{y}{x}$

(example) Determine the polar coordinate.



$$
\begin{aligned}
x^2 + y^2 &= r^2 \\
(1)^2 + (1)^2 &= r^2 \\
1 + 1 &= r^2 \\
2 &= r^2 \\
\sqrt{2} &= r
\end{aligned}
$$

$$
\begin{aligned}
x &= r\cos\theta \\
1 &= \sqrt{2}\cos\theta \\
\frac{1}{\sqrt{2}} &= \cos\theta \\
\frac{\sqrt{2}}{2} &= \cos\theta \\
\theta &= \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \\
\theta &= \frac{\pi}{4}.
\end{aligned}
$$

$= (r, \theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$

(example) Convert the point $(2, \frac{\pi}{3})$ from polar coordinates to cartesian coordinates.

$(2, \frac{\pi}{3}) = (r, \theta)$

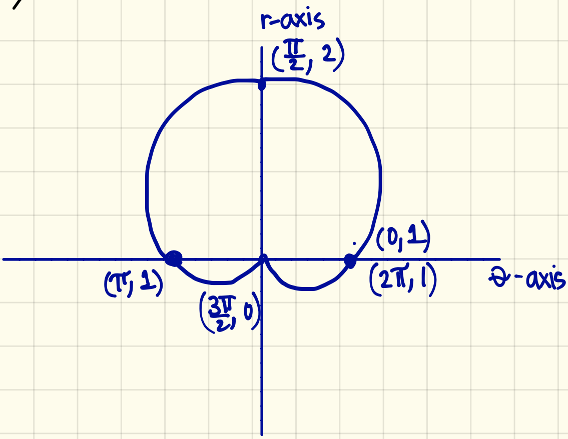
$x = r\cos\theta$
$= 2\cos(\frac{\pi}{3})$
$= 2(\frac{1}{2})$
$= 1.$

$y = r\sin\theta$
$= 2\sin(\frac{\pi}{3})$
$= 2(\frac{\sqrt{3}}{2})$
$= \sqrt{3}.$

$(x, y) = (1, \sqrt{3}).$

[Example] Sketch the curve $r = 1 + \sin\theta$, $0 \le \theta \le 2\pi$.

θ	r
0	$1+\sin 0 = 1$
$\frac{\pi}{2}$	$1+\sin\frac{\pi}{2} = 2$
π	$1+\sin\pi = 1$
$\frac{3\pi}{2}$	$1+\sin\frac{3\pi}{2} = 0$
2π	$1+\sin 2\pi = 1$



[Tangents to Polar Curves]

Let's consider the polar curve $r = f(\theta)$.

Using $x = r\cos\theta$, $y = r\cos\theta$ we can create parametric equations:

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$
$$\;\; = f(\theta)\cos\theta \qquad\qquad \;\; = f(\theta)\sin\theta$$

Let's now differentiate the parametric equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

[Example] For $r = 1 + \sin\theta$, find the slope of the tangent when $\theta = \frac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{(\cos\theta)\sin\theta + (1+\sin\theta)\cos\theta}{(\cos\theta)\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta(1+2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} \frac{\cos\left(\frac{\pi}{3}\right)\left(1+2\sin\left(\frac{\pi}{3}\right)\right)}{\left(1+\sin\left(\frac{\pi}{3}\right)\right)\left(1-2\sin\left(\frac{\pi}{3}\right)\right)} = -1.$$