Formulas and Theorems for Reference

I. Trigonometric Formulas

1. $\sin^2 \theta + \cos^2 \theta = 1$

2. $1 + \tan^2 \theta = \sec^2 \theta$

3. $1 + \cot^2 \theta = \csc^2 \theta$

4. $\sin(-\theta) = -\sin \theta$

5. $\cos(-\theta) = \cos \theta$

6. $\tan(-\theta) = -\tan \theta$

7. $\sin(A + B) = \sin A \cos B + \sin B \cos A$

8. $\sin(A - B) = \sin A \cos B - \sin B \cos A$

9. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

10. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

11. $\sin 2\theta = 2 \sin \theta \cos \theta$

12. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$

13. $\tan \theta = \dfrac{\sin \theta}{\cos \theta} = \dfrac{1}{\cot \theta}$

14. $\cot \theta = \dfrac{\cos \theta}{\sin \theta} = \dfrac{1}{\tan \theta}$

15. $\sec \theta = \dfrac{1}{\cos \theta}$

16. $\csc \theta = \dfrac{1}{\sin \theta}$

17. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin \theta$

18. $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos \theta$

II. Differentiation Formulas

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$

2. $\dfrac{d}{dx}(fg) = fg' + gf'$

3. $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{gf' - fg'}{g^2}$

4. $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

5. $\dfrac{d}{dx}(\sin x) = \cos x$

6. $\dfrac{d}{dx}(\cos x) = -\sin x$

7. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

8. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

9. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

10. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

11. $\dfrac{d}{dx}(e^x) = e^x$

12. $\dfrac{d}{dx}(a^x) = a^x \ln a$

13. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

14. $\dfrac{d}{dx}(\text{Arcsin } x) = \dfrac{1}{\sqrt{1-x^2}}$

15. $\dfrac{d}{dx}(\text{Arctan } x) = \dfrac{1}{1+x^2}$

III. Integration Formulas

1. $\displaystyle\int a\,dx = ax + C$

2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \ n \neq -1$

3. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

4. $\displaystyle\int e^x\,dx = e^x + C$

5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$

6. $\displaystyle\int \ln x\,dx = x\ln x - x + C$

7. $\displaystyle\int \sin x\,dx = -\cos x + C$

8. $\displaystyle\int \cos x\,dx = \sin x + C$

9. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C \ \text{ or } \ -\ln|\cos x| + C$

10. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

12. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$

13. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

14. $\displaystyle\int \sec x\tan x\,dx = \sec x + C$

15. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

16. $\displaystyle\int \csc x\cot x\,dx = -\csc x + C$

17. $\displaystyle\int \tan^2 x\,dx = \tan x - x + C$

18. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\text{Arctan}\left(\frac{x}{a}\right) + C$

19. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Arcsin}\left(\frac{x}{a}\right) + C$

IV. Formulas and Theorems

1. **Limits and Continuity**

    A function $y = f(x)$ is continuous at $x = a$ if:

    i) $f(a)$ is defined (exists)

    ii) $\lim\limits_{x \to a} f(x)$ exists, and

    iii) $\lim\limits_{x \to a} f(x) = f(a)$

    Otherwise, f is discontinuous at $x = a$.

    The limit $\lim\limits_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal — that is,

    $$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

2. **Intermediate-Value Theorem**

    A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

    Note: If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b).

3. **Limits of Rational Functions as $x \to \pm\infty$**

    1. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

        Example: $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x}{x^3 + 3} = 0$

    2. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is infinite if the degree of $f(x) >$ the degree of $g(x)$

        Example: $\lim\limits_{x \to +\infty} \dfrac{x^3 + 2x}{x^2 - 8} = \infty$

    3. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

        Note: The limit will be the ratio of the leading coefficient of $f(x)$ to $g(x)$.

        Example: $\lim\limits_{x \to \infty} \dfrac{2x^2 - 3x + 2}{10x - 5x^2} = -\dfrac{2}{5}$

4.   <u>Horizontal and Vertical Asymptotes</u>

    1. A line $y = b$ is a <u>horizontal asymptote</u> of the graph of $y = f(x)$ if either $\lim\limits_{x \to \infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b$.

    2. A line $x = a$ is a <u>vertical asymptote</u> of the graph of $y = f(x)$ if either $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or $\lim\limits_{x \to a^-} = \pm\infty$.

5.   <u>Average and Instantaneous Rate of Change</u>

    1. <u>Average Rate of Change:</u> If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}.$$

    2. <u>Instantaneous Rate of Change:</u> If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

6. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ or $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

    The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at $x = a$.

    Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

7.   <u>The Number e as a limit</u>

    1. $\lim\limits_{n \to +\infty} \left(1 + \dfrac{1}{n}\right)^n = e$

    2. $\lim\limits_{n \to 0} (1 + n)^{\frac{1}{n}} = e$

8.   <u>Rolle's Theorem</u>

    If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

9.   <u>Mean Value Theorem</u>

    If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

10. <u>Extreme-Value Theorem</u>

    If f is continuous on a closed interval $[a, b]$, then $f(x)$ has both a maximim and a minimum on $[a, b]$.

11. To find the maximum and minimum values of a function $y = f(x)$, locate

    1. the points where $f'(x)$ is zero <u>or</u> where $f'(x)$ fails to exist

    2. the end points, if any, on the domain of $f(x)$.

    Note: These are the <u>only</u> candidates for the value of x where $f(x)$ may have a maximum or a minimum.

12. Let f be differentiable for $a < x < b$ and continuous for $a \leq x \leq b$.

    1. If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.

    2. If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

13. Suppose that $f''(x)$ exists on the interval (a, b).

    1. If $f''(x) > 0$ in (a, b), then f is concave upward in (a, b).

    2. If $f''(x) < 0$ in (a, b), then f is concave downward in (a, b).

    To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ <u>or</u> where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

14. If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does <u>not</u> imply differentiability.

15. <u>Linear Approximation</u>

    The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

    To estimate the slope of a graph at a point — just draw a tangent line to the graph at that point. Another way is (by using a graphics calculator) to "zoom in" around the point in question until the graph "looks" straight. This method almost always works. If we "zoom in" and the graph looks straight at a point, say $x = a$, then the function is <u>locally</u> <u>linear</u> at that point.

    The graph of $y = |x|$ has a sharp corner at $x = 0$. This corner cannot be smoothed out by "zooming in" repeatedly. Consequently, the derivative of $|x|$ does not exist at $x = 0$, hence, is <u>not</u> locally linear at $x = 0$.

16. Comparing Rates of Change

The exponential function $y = e^x$ grows very rapidly as $x \to \infty$ while the logarithmic function $y = \ln x$ grows very slowly as $x \to \infty$.

Exponential functions like $y = 2^x$ or $y = e^x$ grow more rapidly as $x \to \infty$ than polynomial or rational functions, and faster than <u>any</u> power of x, even $x^{100,000}$

Logarithmic functions like $y = \log_2 x$ or $y = \ln x$ grow more slowly as $x \to \infty$ than <u>any</u> positive power of x. The function $y = \ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.

We say, that as $x \to \infty$:

1. $f(x)$ grows <u>faster</u> than $g(x)$ if $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$ or if $\displaystyle\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$.

   If $f(x)$ grows faster than $g(x)$ as $x \to \infty$, then $g(x)$ grows <u>slower</u> than $f(x)$ as $x \to \infty$.

2. $f(x)$ and $g(x)$ grow at the <u>same</u> rate as $x \to \infty$ if $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x \to \infty$ since $\displaystyle\lim_{x \to \infty} \frac{e^x}{x^3} = \infty$

2. x^4 grows faster than $\ln x$ as $x \to \infty$ since $\displaystyle\lim_{x \to \infty} \frac{x^4}{\ln x} = \infty$

3. $x^2 + 2x$ grows at the same rate as x^2 as $x \to \infty$ since $\displaystyle\lim_{x \to \infty} \frac{x^2 + 2x}{x^2} = 1$

To find some of these limits as $x \to \infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

17. Inverse Functions

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g, and, $g(f(x)) = x$, for every x in the domain of f, then, f and g are inverse functions of each other.

2. A function f has an inverse function if and only if no horizontal line intersects its graph more than once.

3. If f is either increasing or decreasing in an interval, then f has an inverse function over that interval.

4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(\text{I})$ and $g'(f(x)) = \dfrac{1}{f'(x)}$.

18. Properties of $y = e^x$

   1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.

   2. The domain is the set of all real numbers, $-\infty < x < \infty$.

   3. The range is the set of all positive numbers, $y > 0$.

   4. $\dfrac{d}{dx}(e^x) = e^x$.

   5. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$.

   6. $y = e^x$ is continuous, increasing, and concave up for all x.

   7. $\lim\limits_{x \to +\infty} e^x = +\infty$ and $\lim\limits_{x \to -\infty} e^x = 0$.

   8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

19. Properties of $y = \ln x$

   1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.

   2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.

   3. $y = \ln x$ is continuous and increasing everywhere on its domain.

   4. $\ln(ab) = \ln a + \ln b$.

   5. $\ln(a/b) = \ln a - \ln b$.

   6. $\ln a^r = r \ln a$.

   7. $y = \ln x < 0$ if $0 < x < 1$.

   8. $\lim\limits_{x \to +\infty} \ln x = +\infty$ and $\lim\limits_{x \to 0^+} \ln x = -\infty$.

   9. $\log_a x = \dfrac{\ln x}{\ln a}$.

20. Trapezoidal Rule

   If a function f is continuous on the closed interval $[a, b]$ where $[a, b]$ has been partitioned into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, each of length $(b - a)/n$, then
   $$\int_a^b f(x)\, dx \approx \frac{b - a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

21. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a, b]$.

1. $\displaystyle\int_a^b c \cdot f(x)\, dx = c \int_a^b f(x)\, dx$ for any constant c

2. $\displaystyle\int_a^a f(x)\, dx = 0$

3. $\displaystyle\int_b^a f(x)\, dx = - \int_a^b f(x)\, dx$

4. $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$, where f is continuous on an interval containing the numbers a, b, and c

5. If $f(x)$ is an odd function, then $\displaystyle\int_{-a}^a f(x)\, dx = 0$

6. If $f(x)$ is an even function, then $\displaystyle\int_{-a}^a f(x)\, dx = 2 \int_0^a f(x)\, dx$

7. If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\, dx \geq 0$

8. If $g(x) \geq f(x)$ on $[a, b]$, then $\displaystyle\int_a^b g(x)\, dx \geq \int_a^b f(x)\, dx$

22a. Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval $[a, b]$. Divide the interval into n equal subintervals, of length $\Delta x = \dfrac{b - a}{n}$. Choose one number in each subinterval i.e. x_1 in the first, x_2 in the second, $\ldots$, x_k in the kth, $\ldots$, and x_n in the nth. Then $\displaystyle\lim_{n \to \infty} \sum_{k=1}^n f(x_k)\Delta x = \int_a^b f(x)\, dx = F(b) - F(a)$.

22b. Fundamental Theorem of Calculus:

$$\int_a^b f(x)\, dx = F(b) - F(a), \text{ where } F'(x) = f(x), \text{ or } \frac{d}{dx}\int_a^x f(x)\, dx = f(x).$$

23. <u>Velocity, Speed, and Acceleration</u>

    1. The <u>velocity</u> of an object tells how fast it is going <u>and</u> in which direction. Velocity is an instantaneous rate of change.

    2. The <u>speed</u> of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.

       The speed of a particle <u>increases</u> (speeds up) when the velocity and acceleration have the same signs. The speed <u>decreases</u> (slows down) when the velocity and acceleration have opposite signs.

    3. The <u>acceleration</u> is the instantaneous rate of change of velocity — it is the derivative of the velocity — that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing. The acceleration gives the rate at which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity $= v(t) = x'(t) = \dfrac{dx}{dt}$

ii) acceleration $= a(t) = x''(t) = v'(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

iii) $v(t) = \displaystyle\int a(t)\, dt$

iv) $x(t) = \displaystyle\int v(t)\, dt$

<u>Note</u>: The <u>average</u> velocity of a particle over the time interval from t_0 to another time t, is

Average Velocity $= \dfrac{\text{Change in position}}{\text{Length of time}} = \dfrac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t.

24.   The average value of $f(x)$ on $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\, dx$.

25.   If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is $\displaystyle\int_a^b [f(x) - g(x)]\, dx$.

26. <u>Volumes of Solids of Revolution</u>

   Let f be nonnegative and continuous on $[a, b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis, and the sides by the lines $x = a$ and $x = b$.

   1. When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \int_a^b \pi[f(x)]^2 \, dx$.

   2. When R is revolved about the y-axis, it generates a solid whose volume $V = \int_a^b 2\pi x f(x) \, dx$.

27. <u>Volumes of Solids with Known Cross Sections</u>

   1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume $= \int_a^b A(x) \, dx$.

   2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume $= \int_c^d A(y) \, dy$.