Fundamental Theorem of Calculus (Solutions)
Student Saturday Session

NATIONAL MATH + SCIENCE INITIATIVE

Multiple Choice Questions: Unless marked otherwise, all questions are no-calc.

1. A
2. C
3. C
4. D
5. A
6. D
7. E
8. A
9. A
10. A
11. E
12. D
13. C
14. C
15. E

Free Response Questions
2005 AB3bc - calc allowed - average score 1.76

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)dx$ in terms of the temperature of the wire.

(b) $\frac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100+93}{2}\cdot 1 + \frac{93+70}{2}\cdot 4 + \frac{70+62}{2}\cdot 1 + \frac{62+55}{2}\cdot 2$$

Average temperature $\approx \frac{1}{8}A = 75.6875°\text{C}$

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45°\text{C}$

The temperature drops $45°\text{C}$ from the heated end of the wire to the other end of the wire.

3 : { 1 : $\frac{1}{8}\int_0^8 T(x)\,dx$; 1 : trapezoidal sum; 1 : answer }

2 : { 1 : value; 1 : meaning }

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2009 AB5b - no calc - average score 2.74

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

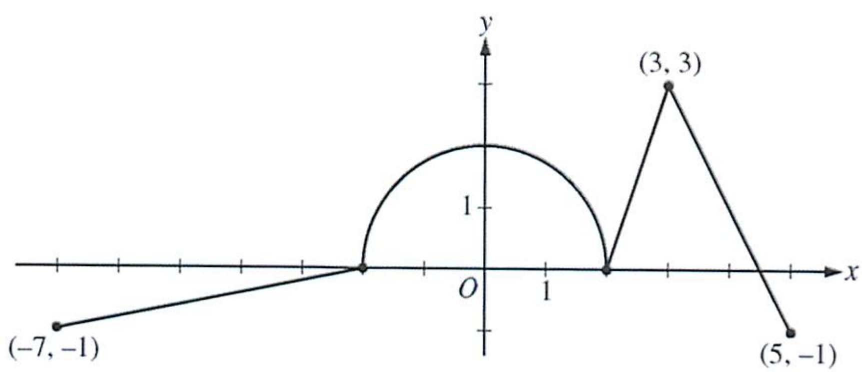
Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(b) Evaluate $\int_2^{13}(3-5f'(x))dx$. Show the work that leads to your answer.

(b) $\int_2^{13}(3-5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$
$= 3(13-2) - 5(f(13) - f(2)) = 8$

2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer }

2010 AB5a - no calc - average score 1.75



Graph of g'

The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find $g(3)$ and $g(-2)$.

(a) $g(3) = 5 + \int_0^3 g'(x)\,dx = 5 + \frac{\pi \cdot 2^2}{4} + \frac{3}{2} = \frac{13}{2} + \pi$

$g(-2) = 5 + \int_0^{-2} g'(x)\,dx = 5 - \pi$

3 : { 1 : uses $g(0) = 5$; 1 : $g(3)$; 1 : $g(-2)$ }

NATIONAL MATH + SCIENCE INITIATIVE

2008B AB4 - no calc

The functions f and g are given by $f(x)=\int_0^{3x}\sqrt{4+t^2}\,dt$ and $g(x)=f(\sin x)$.

(a) Find $f'(x)$ and $g'(x)$.

(b) Write an equation for the line tangent to the graph of $y=g(x)$ at $x=\pi$.

(c) Write, but do not evaluate, an integral expression that represents the maximum value of g on the interval $0\le x\le\pi$. Justify your answer.

Solution	Scoring
(a) $f'(x)=3\sqrt{4+(3x)^2}$ $g'(x)=f'(\sin x)\cdot\cos x$ $=3\sqrt{4+(3\sin x)^2}\cdot\cos x$	$4:\begin{cases}2: f'(x)\\2: g'(x)\end{cases}$
(b) $g(\pi)=0,\ g'(\pi)=-6$ Tangent line: $y=-6(x-\pi)$	$2:\begin{cases}1: g(\pi)\text{ or }g'(\pi)\\1:\text{ tangent line equation}\end{cases}$
(c) For $0<x<\pi,\ g'(x)=0$ only at $x=\frac{\pi}{2}$. $g(0)=g(\pi)=0$ $g\left(\frac{\pi}{2}\right)=\int_0^3\sqrt{4+t^2}\,dt>0$ The maximum value of g on $[0,\pi]$ is $\int_0^3\sqrt{4+t^2}\,dt.$	$3:\begin{cases}1:\text{ sets } g'(x)=0\\1:\text{ justifies maximum at }\frac{\pi}{2}\\1:\text{ integral expression for } g\left(\frac{\pi}{2}\right)\end{cases}$

NATIONAL MATH + SCIENCE INITIATIVE

2009B AB1ac - calc permitted

At a certain height, a tree trunk has a circular cross section. The radius $R(t)$ of that cross section grows at a rate modeled by the function

$$\frac{dR}{dt} = \frac{1}{16}\left(3+\sin\left(t^2\right)\right) \text{ centimeters per year}$$

for $0 \le t \le 3$, where time t is measured in years. At time $t = 0$, the radius is 6 centimeters. The area of the cross section at time t is denoted by $A(t)$.

(a) Write an expression, involving an integral, for the radius $R(t)$ for $0 \le t \le 3$. Use your expression to find $R(3)$

(c) Evaluate $\int_0^3 A'(t)\,dt$. Using appropriate units, interpret the meaning of that integral in terms of cross-sectional area.

(a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1 \text{ miles/minute}^2$	2 : { 1 : answer; 1 : units }
(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.	2 : { 1 : answer; 1 : reason }

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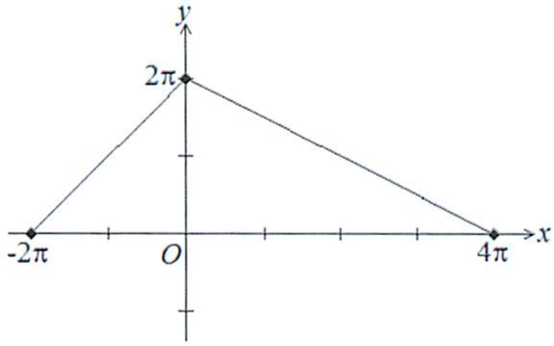
2011B AB6 - no calc



Let g be the piecewise-linear defined function on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.

(a) Find $\int_{-2\pi}^{4\pi} f(x)\,dx$. Show the computations that lead to your answer.

(b) Find all x-values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point.

(c) Let $h(x) = \int_0^{3x} g(t)\,dt$. Find $h'\left(-\frac{\pi}{3}\right)$.

(a) $\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi} \left(g(x) - \cos\left(\frac{x}{2}\right)\right) dx$

$= 6\pi^2 - \left[2\sin\left(\frac{x}{2}\right)\right]_{x=-2\pi}^{x=4\pi}$

$= 6\pi^2$

$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

(b) $f'(x) = g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right) = \begin{cases} 1 + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } -2\pi < x < 0 \\ -\frac{1}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } 0 < x < 4\pi \end{cases}$

$f'(x)$ does not exist at $x = 0$.
For $-2\pi < x < 0$, $f'(x) \neq 0$.
For $0 < x < 4\pi$, $f'(x) = 0$ when $x = \pi$.

f has critical points at $x = 0$ and $x = \pi$.

$4: \begin{cases} 1: \frac{d}{dx}\left(\cos\left(\frac{x}{2}\right)\right) \\ 1: g'(x) \\ 1: x = 0 \\ 1: x = \pi \end{cases}$

(c) $h'(x) = g(3x) \cdot 3$

$h'\left(-\frac{\pi}{3}\right) = 3g(-\pi) = 3\pi$

$3: \begin{cases} 2: h'(x) \\ 1: \text{answer} \end{cases}$